On the physics of the freely falling cats

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All right, let's talk about the falling cats shown in this YouTube video:

https://www.youtube.com/watch?v=RtWbpyjJqrU

The basic idea is the Law of Conservation of Angular Momentum (LCAM) and it works like this (see Fig. 1):

- STAGE 0:

  Split a horizontally suspended cat (with all four legs pointing up) in two halves. Then . . .

- DURING STAGE 1:

  . . . let those halves rotate in opposite directions at different rates.

  You are wondering "How is that possible?" Well, it's possible because Newton's Third Law says that when the two halves interact, they push each other in opposite directions. The pushing in our case is "rotational pushing" and it is happening about the horizontal axis (let's call it the y-axis, see Fig. 1). One half then goes one way around the axis, the other half goes the other way. The role of mass for rotational pushing is played by the so-called moment of inertia, I, so the half with a smaller moment of inertia, I_1^{FRONT} (in our case, the front portion of the cat with *front legs pulled in*), will spin faster (angular velocity $\omega_1^{\mathrm{FRONT}}$) and would rotate by a greater angle $\theta_1^{\mathrm{FRONT}}$ in a time interval Δt_1, while the other half (the back portion of the cat with *hind legs extended*) will spin slower (ω_1^{BACK}) since it has a larger moment of inertia, I_1^{BACK}; it would therefore rotate by a smaller angle θ_1^{BACK} in the opposite direction.

- DURING STAGE 2:

  The two halves switch roles and are now *pulling* (instead of pushing) each other. [This is needed to get the cat back to its normal shape, for we don't want the cat to remain twisted.] This time, the cat has its *front legs extended*, so its front half's moment of inertia I_2^{FRONT} is larger and is spinning at a slower rate (angular velocity $\omega_2^{\mathrm{FRONT}}$), thus covering a small angle $\theta_2^{\mathrm{FRONT}}$ around the y-axis in time Δt_2. The back of the cat, on the other hand, has the *hind legs pulled in*, making its moment of inertia I_2^{BACK} smaller; this half spins faster (with ω_2^{BACK}) and covers a greater angle θ_2^{BACK} in time Δt_2.

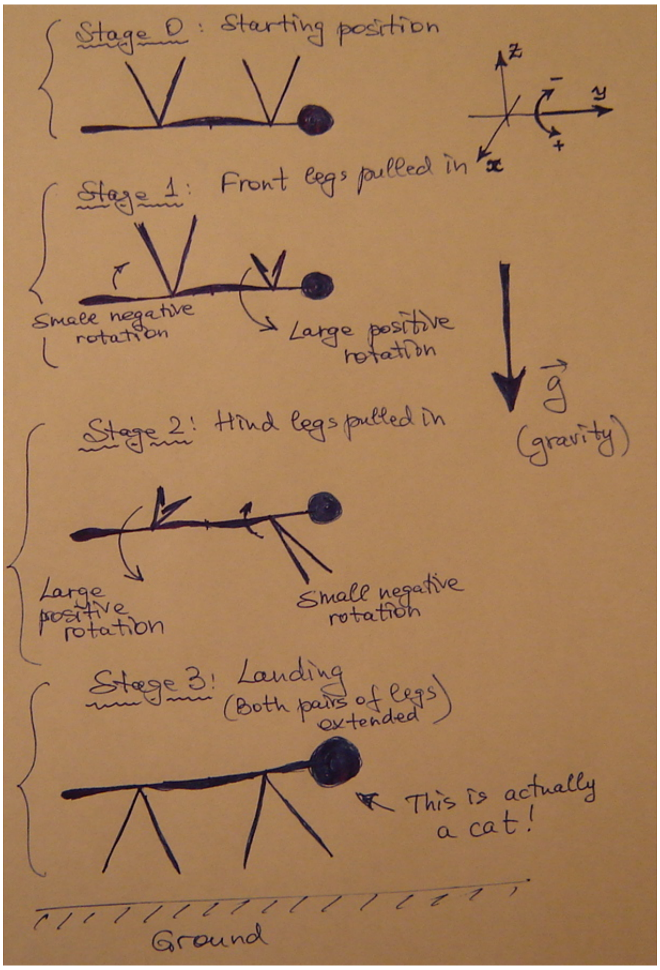

FIG. 1: A freely falling cat landing on its feet.

- STAGE 3:

  Once the body untwists, the legs will face the ground, making the cat fully prepared for soft landing.

Now let's do a bit of Math. For simplicity, let's assume that Stages 1 and 2 last for a same time,

$$\Delta t_1 = \Delta t_2 \equiv \Delta t, \tag{1}$$

and that

$$\begin{aligned} I_1^{\mathrm{BACK}} = I_2^{\mathrm{FRONT}} &\equiv I_0, & (2)\\ I_1^{\mathrm{FRONT}} = I_2^{\mathrm{BACK}} &\equiv kI_0, \quad 0 < k < 1, & (3)\\ \theta_1^{\mathrm{FRONT}} + \theta_2^{\mathrm{FRONT}} &= \pi, & (4)\\ \theta_1^{\mathrm{BACK}} + \theta_2^{\mathrm{BACK}} &= \pi, & (5) \end{aligned}$$

where I_0 is the maximal possible moment of inertia of either half of the cat's body (with the corresponding pair of legs stretched out, making a figure Γ with the body) and k is the factor by which I_0 can be reduced (say, by pulling the said pair of legs in). Then the LCAM says that the total angular momentum of the cat stays zero,

$$\begin{aligned} kI_0\omega_1^{\rm FRONT} + I_0\omega_1^{\rm BACK} &= 0 \quad \text{(STAGE 1)}, \qquad (6)\\ I_0\omega_2^{\rm FRONT} + kI_0\omega_2^{\rm BACK} &= 0 \quad \text{(STAGE 2)}, \qquad (7)\end{aligned}$$

and Eqs. (4) and (5) say that both halves have been rotated (in time $2\Delta t$) by the same positive angle π,

$$\begin{aligned} \omega_1^{\rm FRONT}\Delta t + \omega_2^{\rm FRONT}\Delta t &= \pi, \qquad (8)\\ \omega_1^{\rm BACK}\Delta t + \omega_2^{\rm BACK}\Delta t &= \pi. \qquad (9)\end{aligned}$$

Solving the system (6), (7), (8), (9), gives

$$\begin{aligned} \theta_1^{\rm FRONT} &\equiv \omega_1^{\rm FRONT}\Delta t = \frac{\pi}{1-k}, \qquad (10)\\ \theta_1^{\rm BACK} &\equiv \omega_1^{\rm BACK}\Delta t = -\frac{k\pi}{1-k}, \qquad (11)\\ \theta_2^{\rm FRONT} &\equiv \omega_2^{\rm FRONT}\Delta t = -\frac{k\pi}{1-k}, \qquad (12)\\ \theta_2^{\rm BACK} &\equiv \omega_2^{\rm BACK}\Delta t = \frac{\pi}{1-k}. \qquad (13)\end{aligned}$$

We see that in order to make the twisting process "comfortable" for the cat, the k-parameter must be small (ideally, zero), otherwise the angles by which the halves twist will be much greater than π. To achieve this, the cat must pull its legs (the front legs during Stage 1 and the hind legs during Stage 2) very close to its body (possibly even extending them *horizontally*!).

Anyway, here's an example. For $k = 1/4$, we get

$$\theta_1^{\rm FRONT} = \frac{4\pi}{3}, \quad \theta_1^{\rm BACK} = -\frac{\pi}{3}, \qquad (14)$$

$$\theta_2^{\rm FRONT} = -\frac{\pi}{3}, \quad \theta_2^{\rm BACK} = \frac{4\pi}{3}, \qquad (15)$$

which is OK for a typical cat, but not OK for a typical human.

PS: For cats trying to use their twisting skills in microgravitational environment see (starting at 3:30 min.) the following link:

https://www.youtube.com/watch?v=HwRdcv8azvk

PPS: Dr. Robin Shelton pointed out a curious upward hind-leg extension during Stage 1, and the Λ-shaped body upon landing. I think those are due to the Law of Conservation of *Linear* Momentum, which requires the Center of Mass not to move (gravity aside). So if during Stage 1 the front legs are being pulled in and rotated downward, something must go up in order to keep the CoM in place. The cat chooses to do that by extending one hind leg even higher (which, incidentally, helps to decrease $|\theta_1^{\rm BACK}|$ too!).

During Stage 2, when the hind legs rotate downwards, the stability of the CoM is achieved by bending the body upwards. That seems to be the main reason for the cat's Λ-shape at landing.