The University of Georgia
Department of Physics and Astronomy

Prelim Exam
January 5, 2024

Part I (problems 1, 2, 3, and 4)
9:00 am – 1:00 pm

Instructions:

- Start each problem on a new sheet of paper. Write the problem number on the top left of each page and your pre-arranged prelim ID number (but ***not*** your name) on the top right of each page.
- Leave margins for stapling and photocopying.
- Write only on *one side* of the paper. Please ***do not*** write on the back side.
- If not advised otherwise, derive the mathematical solution for a problem from basic principles or general laws (Newton's laws, the Maxwell equations, the Schrödinger equation, etc.).
- You may use a calculator for basic operations only (i.e., not for referring to notes stored in memory, symbolic algebra, symbolic and numerical integration, etc.) The use of cell phones, tablets, and laptops is not permitted.
- Show your work and/or explain your reasoning in *all* problems, as the graders are not able to read minds. Even if your final answer is correct, not showing your work and reasoning will result in a *substantial* penalty.
- Write your work and reasoning in a neat, clear, and logical manner so that the grader can follow it. Lack of clarity is likely to result in a substantial penalty.

Problem 1: Classical Mechanics (CM1)

A flexible rope of length $L = 1.0\ m$ slides from a frictionless table top as shown in the figure below. The rope is initially released from rest, with a length $x = 20\ cm$ hanging over the edge of the table.

1. What is the equation of motion?
2. Find the time at which the left end of the rope reaches the edge of the table.

Solution CM1:

A flexible rope of length $L = 1.0$ m slides from a frictionless table top as shown in Figure 1.
The rope is initially released from rest, with a length $x = 30$ cm hanging over the edge of the table.

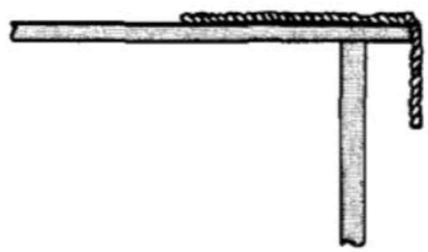


Figure 1: Diagram of rope arranged on table.

1. What is the equation of motion?
2. Find the time at which the left end of the rope reaches the edge of the table.

1.2 Solution to 9-21

1. The equation of motion is:

$$\frac{mgx}{L} = m\frac{dx^2}{dt^2} \Rightarrow \ddot{x} = \frac{gx}{L} \tag{1}$$

2. We need a solution of the form:

$$x = Ae^{\omega t} + Be^{-\omega t}$$

Putting this into equation of motion, we find

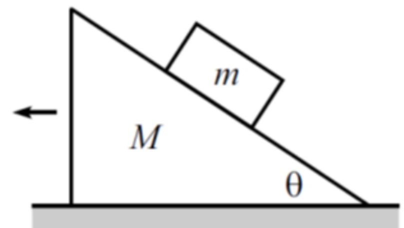
$$\omega = \sqrt{\frac{g}{L}}$$

Initial conditions are $x_{(t=0)} = x_0 = 0.3$ m; $v_{(t=0)} = 0$ m/s. From these we find $A = B = x_0/2$. Finally $x = x_0 \cosh(\omega t)$. When $x = L$, the corresponding time is

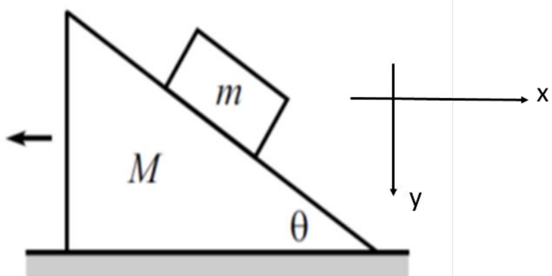
$$t = \frac{1}{\omega}\cosh^{-1}\left(\frac{L}{x_0}\right) = 0.59 \text{ s}$$

Problem 2: Classical Mechanics (CM2)

A block of mass *m* is held motionless on the frictionless inclined plane of a wedge, mass *M* and angle of inclination θ (see the figure). The wedge rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the wedge?



Hint: You need an x- and a y-coordinates for the block and an x-coordinate for the wedge. These three coordinates are subject to a constraint equation, such that the block is always in contact with the inclined surface of the wedge.



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Solution CM2:

Let N be the normal force between the block and the plane. Note that we *cannot* assume that $N = mg\cos\theta$, because the plane recoils. We can see that $N = mg\cos\theta$ is in fact incorrect, because in the limiting case where $M = 0$, we have no normal force at all.

The various $F = ma$ equations (vertical and horizontal for the block, and horizontal for the plane) are

$$\begin{aligned} mg - N\cos\theta &= ma_y, \\ N\sin\theta &= ma_x, \\ N\sin\theta &= MA_x, \end{aligned} \tag{2.59}$$

where we have chosen the positive directions for a_y, a_x, and A_x to be downward, rightward, and leftward, respectively. There are four unknowns here: a_x, a_y, A_x, and N. So we need one more equation. This fourth equation is the constraint that the block remains in contact with the plane. The horizontal distance between the block and its starting point on the plane is $(a_x+A_x)t^2/2$, and the vertical distance is $a_y t^2/2$. The ratio of these distances must equal $\tan\theta$ if the block is to remain on the plane. Therefore, we must have

$$\frac{a_y}{a_x + A_x} = \tan\theta. \tag{2.60}$$

Using eqs. (2.59), this becomes

$$\frac{g - \frac{N}{m}\cos\theta}{\frac{N}{m}\sin\theta + \frac{N}{M}\sin\theta} = \tan\theta$$

$$\Longrightarrow \qquad N = g\left(\sin\theta\tan\theta\left(\frac{1}{m} + \frac{1}{M}\right) + \frac{\cos\theta}{m}\right)^{-1}. \tag{2.61}$$

(In the limit $M \to \infty$, this reduces to $N = mg\cos\theta$, as it should.) Having found N, the third of eqs. (2.59) gives A_x, which may be written as

$$A_x = \frac{N\sin\theta}{M} = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}. \tag{2.62}$$

Problem 3: Electromagnetism (EM1)

Calculate the electric field $\boldsymbol{E}$ at the point P in a distance z above the midpoint of a straight wire of length $2L$ with constant line charge density λ (thus, the total charge is $Q = 2\lambda L$). For symmetry reasons, it is convenient for the calculation to choose the midpoint of the wire as the origin of the coordinate system and to decompose the problem into components parallel and perpendicular to the wire (why?).

Discuss the limiting cases $z \gg L$ and $L \to \infty$.

Hint: You may want to use this integral formula:

$$\int_0^L dx' \, \frac{1}{(x'^2 + z^2)^{3/2}} = \frac{L}{z^2\sqrt{L^2 + z^2}}$$

Solution EM1:

Generally, the electric field of a charged contour is given by:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_C dr' \, \frac{\lambda(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} (\boldsymbol{r} - \boldsymbol{r}').$$

Here, it simplifies to:

$$\boldsymbol{E}^{\mathrm{P}} = \frac{\lambda}{4\pi\epsilon_0} \int_{-L}^{L} dx' \, \frac{1}{|\boldsymbol{l}(x')|^3} \boldsymbol{l}(x'),$$

where $\boldsymbol{l}(x')$ is the vector from any point of the wire to P.

Use decomposition: $\boldsymbol{l} = \boldsymbol{l}_{\parallel} + \boldsymbol{l}_{\perp} = -x'\boldsymbol{e}_x + z\boldsymbol{e}_z$ ($\boldsymbol{e}_x$ and $\boldsymbol{e}_z$ are unit vectors parallel and perpendicular to the wire, respectively).

Since $|\boldsymbol{l}(x')| = \sqrt{x'^2 + z^2}$,

$$\boldsymbol{E}_{\parallel}^{\mathrm{P}} = -\boldsymbol{e}_x \frac{\lambda}{4\pi\epsilon_0} \int_{-L}^{L} dx' \, \frac{x'}{(x'^2 + z^2)^{3/2}} = 0 \quad \text{(symmetry)}$$

$$\boldsymbol{E}_{\perp}^{\mathrm{P}} = \boldsymbol{e}_z \frac{\lambda z}{4\pi\epsilon_0} \int_{-L}^{L} dx' \, \frac{1}{(x'^2 + z^2)^{3/2}} = \frac{Q}{4\pi\epsilon_0} \frac{1}{z\sqrt{z^2 + L^2}} \boldsymbol{e}_z, \qquad Q = 2\lambda L$$

Limits:

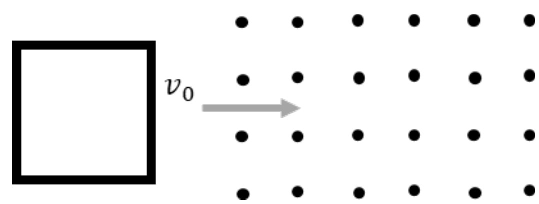
$z \gg L$: $\boldsymbol{E}_{\perp}^{\mathrm{P}} = \frac{Q}{4\pi\epsilon_0} \frac{1}{z^2} \boldsymbol{e}_z$ (finite wire looks like a point charge from the distance)

$L \to \infty$: $\boldsymbol{E}_{\perp}^{\mathrm{P}} = \frac{2\lambda}{4\pi\epsilon_0} \frac{1}{z} \boldsymbol{e}_z$ (field of an infinite wire)

Problem 4: Electromagnetism (EM2)

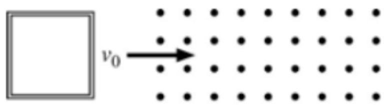
A square loop, length l on each side, moves with velocity $v_0$into a uniform magnetic field *B*, which is perpendicular to the plane of the loop. The loop has mass m and resistance of R, and it enters the field at t = 0 s. Assume that the loop is moving to the right along the x-axis and the field begins at $x = 0\ m$.

Assuming the back edge of the loop has not enter the field and ignoring gravity, find an expression for the loop's velocity as a function of time as it enters the magnetic field.

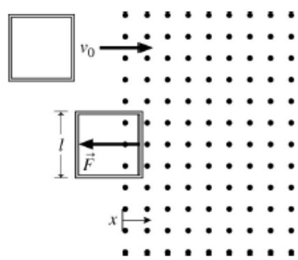


Solution EM2:

EM1. A square loop, length l on each side, moves with velocity v_0 into a uniform magnetic field B, which is perpendicular to the plane of the loop. The loop has mass m and resistance of R, and it enters the field at $t=0$ s. Assume that the loop is moving to the right along the x-axis and the field begins at $x=0$ m.
Assuming the back edge of the loop has not entered the field and ignoring gravity, find an expression for the loop's velocity as a function of time as it enters the magnetic field.



Solve: Assume the field changes abruptly at the boundary and is constant.



The loop moving through the field will have a changing flux. This will produce an induced emf and corresponding current. The induced current will oppose the change in flux and result in a retarding force on the loop.

The loop moving into the field will have an increasing outward flux. The induced current will oppose this change and create a field that is into the page. This requires that the induced current flow clockwise. The current-carrying wires of the loop will experience a force. The force on the leading edge is to the left, retarding the motion. The forces on the top and bottom edges cancel and the trailing edge experiences no force since the field is zero there. The induced current is

$$I = \frac{\mathcal{E}}{R} = \frac{1}{R}\left|\frac{d\Phi}{dt}\right| = \frac{B}{R}\frac{dA}{dt} = \frac{Bl}{R}\frac{dx}{dt} = \frac{Blv}{R}$$

With this current we can find the force on the leading edge. Remembering that it opposes the motion, we have

$$F = IlB = \left(\frac{Blv}{R}\right)lB = \left(\frac{B^2l^2}{R}\right)v$$

Newton's second law is

$$ma = m\frac{dv}{dt} = -\left(\frac{B^2l^2}{R}\right)v \;\Rightarrow\; \frac{dv}{dt} = -\left(\frac{B^2l^2}{mR}\right)v \;\Rightarrow\; v = v_0e^{-bt}$$

where $b = l^2B^2/(mR)$ and the initial velocity is v_0.

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Department of Physics and Astronomy

Prelim Exam
January 5, 2024

Part II (problems 5 and 6)
3:00 pm – 5:00 pm

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Problem 5: Quantum Mechanics (QM1)

Consider the quintessential introduction quantum mechanics problem of one dimensional simple harmonic oscillator (SHO) with mass m and spring constant k, described by a Hamiltonian operator $\hat{H}$ and a wavefunction ψ.

1. What is the SHO angular oscillation frequency, ω, in terms of m and k? What is k in term of ω and m?
2. Write down the Hamiltonian operator $\hat{H}$ for the SHO in terms of the momentum operator $\hat{p}$ and the position operator $\hat{x}$.
3. What is the commutator: $[\hat{x}, \hat{p}]$= ?
   Next, consider the ladder operators:

$$\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p})$$
$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p})$$

4. Show:

$$\hat{H} = \hbar\omega(\hat{a}_-\hat{a}_+ - \frac{1}{2})$$

   Must give details to receive full credit.
5. Evaluate the commutator: $[\hat{a}_-, \hat{a}_+]$ =?

Solution QM1:

1. $\omega = \sqrt{\frac{k}{m}}$, $k = m\omega^2$.
2. $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{1}{2m}(\hat{p}^2 + (m\omega\hat{x})^2)$
3. $[\hat{x}, \hat{p}] = i\hbar$
4. First evaluate: with $[\hat{x}, \hat{p}] = i\hbar$

$$\hat{a}_-\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p})\frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p}) = \frac{1}{2\hbar m\omega}((m\omega\hat{x})^2 + \hat{p}^2 - im\omega[\hat{x}, \hat{p}])$$

$$= \frac{1}{2\hbar m\omega}((m\omega\hat{x})^2 + \hat{p}^2 + \hbar m\omega) = \frac{\hat{H}}{\hbar\omega} + \frac{1}{2} \quad => \quad \hat{H} = \hbar\omega(\hat{a}_-\hat{a}_+ - \frac{1}{2})$$

5. Also evaluate:

$$\hat{a}_+\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p})\frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p}) = \frac{1}{2\hbar m\omega}((m\omega\hat{x})^2 + \hat{p}^2 + im\omega[\hat{x}, \hat{p}])$$

$$= \frac{1}{2\hbar m\omega}((m\omega\hat{x})^2 + \hat{p}^2 - \hbar m\omega) = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$$

$$=> [\hat{a}_-, \hat{a}_+] = 1$$

Problem 6: Quantum Mechanics (QM2)

The wave function $\psi(x,t)$ of quantum particle moving along the x-axis obeys the time-dependent Schrödinger equation

[1] $$i\hbar\partial_t\psi(x,t) = H\psi(x,t)$$

where ∂_t denotes the partial derivative with respect to time t and H is the Hamiltonian operator. H acts only on the x-dependence of $\psi(x,t)$ and H is a Hermitian operator which means that

[2] $$\int_{-\infty}^{+\infty} \chi(x)^* \, \big(H\phi(x)\big) \, dx = \int_{-\infty}^{+\infty} (H\chi(x))^* \, \phi(x) \, dx$$

for any two normalizable, complex-valued wave functions $\phi(x)$ and $\chi(x)$. Here, Z^* denotes the complex conjugate of Z, for any complex number Z.

Let $N_\psi(t)$ denote the norm of the wave function $\psi(x,t)$ at time t, defined by

$$N_\psi(t) = \int_{-\infty}^{+\infty} |\psi(x,t)|^2 \, dx$$

Use only Eqs. [1] and [2] to prove that $N_\psi(t)$ is time-independent.

Hints: (1) Recall that $|Z|^2 = Z^*Z$, for any complex number Z. (2) The time-independence of $N_\psi(t)$ can be equivalently stated in terms of its time derivative, $\partial_t N_\psi(t)$. In other words, what is $\partial_t N_\psi(t)$ if and only if $N_\psi(t)$ is independent of time?

Solution QM2:

Using Hint (1), write

$$N_{\psi}(t) = \int_{-\infty}^{+\infty} \left(\psi(x,t)\right)^{*}\psi(x,t) \ \ dx$$

Then, following Hint (2), try to evaluate $\partial_t N_{\psi}(t)$:

$$\begin{aligned}\partial_t N_{\psi}(t) &= \partial_t\left(\int_{-\infty}^{+\infty} \left(\psi(x,t)\right)^{*}\psi(x,t) \ \ dx\right) \\ &= \int_{-\infty}^{+\infty} \partial_t\left(\left(\psi(x,t)\right)^{*}\psi(x,t)\right) \ dx \\ &= \int_{-\infty}^{+\infty} \left(\left(\partial_t\psi(x,t)\right)^{*}\psi(x,t) + \left(\psi(x,t)\right)^{*}\left(\partial_t\psi(x,t)\right)\right) \ dx\end{aligned}$$

Using Eq.[1], get

$$\partial_t\psi(x,t) = -\frac{i}{\hbar} H\psi(x,t)$$

$$\left(\partial_t\psi(x,t)\right)^{*} = \frac{i}{\hbar}\left(H\psi(x,t)\right)^{*}$$

Insert this into above eq. for $\partial_t N_{\psi}(t)$:

$$\begin{aligned}\partial_t N_{\psi}(t) &= \int_{-\infty}^{+\infty} \left(\frac{i}{\hbar}\left(H\psi(x,t)\right)^{*}\psi(x,t) - \frac{i}{\hbar}\left(\psi(x,t)\right)^{*}\left(H\psi(x,t)\right)\right) \ dx \\ &= \frac{i}{\hbar}\left(\int_{-\infty}^{+\infty}\left(H\psi(x,t)\right)^{*}\psi(x,t)\, dx \ - \int_{-\infty}^{+\infty}\left(\psi(x,t)\right)^{*}\left(H\psi(x,t)\right) dx \right)\end{aligned}$$

Using Eq.[2], the first integral in the preceding line is the same as the second integral:

$$\int_{-\infty}^{+\infty}\left(H\psi(x,t)\right)^{*}\psi(x,t)\, dx = \int_{-\infty}^{+\infty}\left(\psi(x,t)\right)^{*}\left(H\psi(x,t)\right) dx$$

Therefore

$$\begin{aligned}\partial_t N_{\psi}(t) &= \frac{i}{\hbar}\left(\int_{-\infty}^{+\infty}\left(\psi(x,t)\right)^{*}\left(H\psi(x,t)\right) dx \ - \int_{-\infty}^{+\infty}\left(\psi(x,t)\right)^{*}\left(H\psi(x,t)\right) dx \right) \\ &= 0\end{aligned}$$

at all times t. But if $\partial_t N_{\psi}(t) = 0$ at all times t then $N_{\psi}(t)$ must be a constant in t, *i.e.,* $N_{\psi}(t)$ must be independent of t. *Q.E.D.*