

**The University of Georgia**  
**Department of Physics and Astronomy**

**Prelim Exam**  
**January 9, 2026**

**Part II (Problems 5 and 6)**  
**3:00 pm – 5:00 pm**

**Instructions:**

- Start each problem on a new sheet of paper. Write the problem number on the top left of each page and your pre-arranged prelim ID number (but *not* your name) on the top right of each page.
- Leave margins for stapling and photocopying.
- Write only on *one side* of the paper. Please *do not* write on the back side.
- If not advised otherwise, derive the mathematical solution for a problem from basic principles or general laws (Newton's laws, the Maxwell equations, the Schrödinger equation, *etc.*).
- You may use a calculator for basic operations only (i.e., not for referring to notes stored in memory, symbolic algebra, symbolic and numerical integration, etc.) The use of cell phones, tablets, and laptops is not permitted.
- Show your work and/or explain your reasoning in *all* problems, as the graders are not able to read minds. Even if your final answer is correct, not showing your work and reasoning will result in a *substantial* penalty.
- Write your work and reasoning in a neat, clear, and logical manner so that the grader can follow it. Lack of clarity is likely to result in a substantial penalty.

## Problem 5: Quantum Mechanics (QM 1)

Set  $\hbar = m = 1$  and consider two Hamiltonians,

$$H_- = A^\dagger A, \quad H_+ = AA^\dagger,$$

where

$$A = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + x \right), \quad A^\dagger = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + x \right).$$

### Part 1: Schrödinger form

Use  $A$  and  $A^\dagger$  given above to write  $H_-$  and  $H_+$  explicitly in the Schrödinger form,

$$H_- = -\frac{1}{2} \frac{d^2}{dx^2} + V_-(x), \quad H_+ = -\frac{1}{2} \frac{d^2}{dx^2} + V_+(x).$$

Identify the corresponding potentials  $V_-(x)$  and  $V_+(x)$ .

### Part 2: Spectra

By comparing to the standard harmonic oscillator Hamiltonian,

$$H_{\text{osc}} = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 \right), \quad E_n^{\text{osc}} = n + \frac{1}{2}, \quad n = 0, 1, 2, \dots,$$

determine the spectra  $E_n^-$  and  $E_n^+$  of  $H_-$  and  $H_+$ .

### Part 3: Pairing of positive-energy states via $A$ and $A^\dagger$

(a) If  $\varphi_n^-$  is an eigenstate of  $H_-$  with positive eigenvalue  $E_n > 0$ , show that  $\varphi_n^+$ , defined by

$$\varphi_n^+ = \frac{1}{\sqrt{E_n}} A \varphi_n^-,$$

is an eigenstate of  $H_+$  with the same eigenvalue.

(b) Similarly, if  $\varphi_n^+$  is an eigenstate of  $H_+$  with positive eigenvalue  $E_n > 0$ , show that

$$\varphi_n^- = \frac{1}{\sqrt{E_n}} A^\dagger \varphi_n^+$$

is an eigenstate of  $H_-$  with the same eigenvalue.

### Problem 6: Quantum Mechanics (QM 2)

A particle of mass  $m$  is confined to a one-dimensional infinite potential well,

$$U(x) = \begin{cases} +\infty, & x \leq 0, \\ 0, & 0 < x < a, \\ +\infty, & x \geq a. \end{cases}$$

The particle is prepared in the state  $\psi$  with wavefunction

$$\psi(x) = -Cx(x - a),$$

where  $C > 0$  is the normalization constant. Work in standard SI units.

- a) Find the value of  $C$  that normalizes  $\psi(x)$ .
- b) Find the average energy  $\bar{E}$  of the particle in the state  $\psi(x)$ .
- c) Find the ground-state energy  $E_{\text{ground}}$  of the system and compare it with  $\bar{E}$  obtained in part b). Are the two energies close or noticeably different? Explain.

## Solution to Problem 5: Quantum Mechanics (QM 1)

### Part 1: Schrödinger form

Compute  $H_- = A^\dagger A$ :

$$A^\dagger A = (1/2)(-d/dx + x)(d/dx + x) = (1/2)[-d^2/dx^2 - (d/dx)x + x(d/dx) + x^2]$$

When acting on  $\psi(x)$ , use the product rule to eliminate first-order terms, giving,

$$H_- = (1/2)(-d^2/dx^2 + x^2 - 1)$$

Similarly,

$$H_+ = (1/2)(-d^2/dx^2 + x^2 + 1)$$

Thus, the potentials are

$$V_-(x) = (1/2)(x^2 - 1), \quad V_+(x) = (1/2)(x^2 + 1).$$

### Part 2: Spectra

The standard harmonic oscillator has  $H_{\text{osc}} = (1/2)(-d^2/dx^2 + x^2)$ , with  $E_n^{\text{osc}} = n + 1/2$ .

Therefore:

$$H_- = H_{\text{osc}} - 1/2 \rightarrow E_n^- = n, \quad n = 0, 1, 2, \dots$$

$$H_+ = H_{\text{osc}} + 1/2 \rightarrow E_n^+ = n + 1, \quad n = 0, 1, 2, \dots$$

or,

$$E_n^- = 0, 1, 2, 3, \dots; \quad E_n^+ = 1, 2, 3, \dots$$

### Part 3: Pairing of positive-energy states via $A$ and $A^\dagger$

For  $E_n > 0$ , assume  $H_- \varphi_n^- = E_n \varphi_n^-$  and define  $\varphi_n^+ = \frac{1}{\sqrt{E_n}} A \varphi_n^-$ . Then check the eigenvalue equation:

$$H_+ \varphi_n^+ = A A^\dagger \frac{1}{\sqrt{E_n}} A \varphi_n^- = A \frac{1}{\sqrt{E_n}} A^\dagger A \varphi_n^- = A \frac{1}{\sqrt{E_n}} H_- \varphi_n^- = A \frac{1}{\sqrt{E_n}} E_n \varphi_n^- = E_n \varphi_n^+.$$

Similarly, for  $\varphi_n^- = \frac{1}{\sqrt{E_n}} A^\dagger \varphi_n^+$ ,

$$H_- \varphi_n^- = A^\dagger A \frac{1}{\sqrt{E_n}} A^\dagger \varphi_n^+ = A^\dagger \frac{1}{\sqrt{E_n}} A A^\dagger \varphi_n^+ = A^\dagger \frac{1}{\sqrt{E_n}} H_+ \varphi_n^+ = A^\dagger \frac{1}{\sqrt{E_n}} E_n \varphi_n^+ = E_n \varphi_n^-.$$

Thus  $A$  and  $A^\dagger$  map positive-energy eigenstates of  $H_-$  and  $H_+$  into one another.

(NOTE: The normalization factor  $(1/\sqrt{E_n})$  ensures  $\langle \varphi | \varphi \rangle = 1$  for both.)

**REMARKS:** In supersymmetric quantum mechanics (SUSY QM), one introduces *partner Hamiltonians*,

$$H_- = A^\dagger A, \quad H_+ = A A^\dagger,$$

connected via differential operators,

$$A = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + W(x) \right), \quad A^\dagger = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + W(x) \right),$$

called *supercharges*, where  $W(x) = W^\dagger(x)$  is the *superpotential*. This allows exploration of energy-level pairing (as we did above) and related SUSY features. Here we considered the simplest model with superpotential  $W(x) = x$ . In this model, positive-energy states  $E_n^+ = E_{n+1}^- = 1, 2, 3, \dots$ , are paired. The unpaired  $E_0^- = 0$  state  $\varphi_0^-$  of  $H_-$  satisfying  $A\varphi_0^- = 0$  (which gives  $\varphi_0^-(x) = Ce^{-x^2/2}$ , as can be checked by direct calculation) represents the *supersymmetric vacuum*. When such *normalizable* zero-energy ground state exists, SUSY is said to be *unbroken*. If no normalizable zero-energy state exists in either Hamiltonian, the spectra of  $H_-$  and  $H_+$  become identical (with the lowest energy states of both Hamiltonians being strictly positive), and SUSY is said to be *broken*.

## Solution to Problem 6: Quantum Mechanics (QM 2)

### a) Normalization

We require

$$\int_0^a |\psi(x)|^2 dx = 1.$$

Substituting

$$\psi(x) = -Cx(x - a),$$

we get,

$$\begin{aligned} \int_0^a |\psi(x)|^2 dx &= C^2 \int_0^a x^2(x - a)^2 dx = C^2 \int_0^a (x^4 - 2ax^3 + a^2x^2) dx = C^2 \left( \frac{x^5}{5} - \frac{2ax^4}{4} + \frac{a^2x^3}{3} \right) \Big|_0^a \\ &= C^2 a^5 \left( \frac{1}{5} - \frac{1}{2} + \frac{1}{3} \right) = \frac{C^2 a^5}{30} = 1, \end{aligned}$$

or,

$$C = \sqrt{\frac{30}{a^5}}.$$

### b) Average energy (with $\psi(x)$ normalized)

The average energy is

$$\bar{E} = \int_0^a \psi^*(x) \hat{H} \psi(x) dx,$$

where for a free particle inside the well,

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}.$$

We have,

$$\begin{aligned} \bar{E} &= C^2 \int_0^a x(x - a) \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \right) [x(x - a)] dx = C^2 \int_0^a x(x - a) \left( -\frac{\hbar^2}{2m} \times 2 \right) dx = \\ &= -\frac{C^2 \hbar^2}{m} \int_0^a x(x - a) dx = -\frac{C^2 \hbar^2}{m} \left( \frac{x^3}{3} - \frac{ax^2}{2} \right) \Big|_0^a = -\frac{C^2 \hbar^2 a^3}{m} \left( \frac{1}{3} - \frac{1}{2} \right) = \frac{C^2 \hbar^2 a^3}{6m}. \end{aligned}$$

Substituting  $C^2 = 30/a^5$  gives

$$\bar{E} = \frac{5\hbar^2}{ma^2}.$$

### c) Comparison with the Ground State

For the infinite well, the normalized ground-state wavefunction and energy are

$$\psi_{\text{ground}}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right), \quad E_{\text{ground}} = \frac{\pi^2 \hbar^2}{2ma^2}.$$

These can be found by solving the corresponding Schrödinger equation,

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi_k(x)}{dx^2} = E_k \psi_k(x), \quad \psi_k \sim \sin(kx),$$

with zero boundary conditions and selecting the nontrivial solution with lowest energy ( $k_{\text{ground}} = \pi/a$ ). Numerically,

$$\frac{\pi^2}{2} \approx 4.93,$$

so

$$E_{\text{ground}} \approx 4.93 \frac{\hbar^2}{ma^2}, \quad \bar{E} = 5 \frac{\hbar^2}{ma^2}.$$

Thus,

$$\bar{E} \approx E_{\text{ground}}.$$

This close agreement can be understood by comparing the shapes of the two wavefunctions. Both  $\psi(x)$  (inverted parabola) and  $\psi_{\text{ground}}(x)$  (sine function) are concave downward in the interval  $0 < x < a$ , both are “symmetric” with respect to  $x = a/2$ , and both go to zero at the endpoints. In addition, both wavefunctions are normalized to 1. Because of these similarities, their shapes must be close to each other. Correspondingly, their overlap (the integral of their product) is nearly 1. This means that when we expand  $\psi(x)$  as a sum of energy eigenstates, the ground-state wavefunction contributes the most. As a result, the average energy of the system is dominated by the ground-state energy, which explains why  $\bar{E} \approx E_{\text{ground}}$ . The following figure confirms our intuition:

