The University of Georgia
Department of Physics and Astronomy

Prelim Exam
January 9, 2026

Part I (Problems 1, 2, 3, and 4)
9:00 am – 1:00 pm

Instructions:

- Start each problem on a new sheet of paper. Write the problem number on the top left of each page and your pre-arranged prelim ID number (but ***not*** your name) on the top right of each page.
- Leave margins for stapling and photocopying.
- Write only on *one side* of the paper. Please ***do not*** write on the back side.
- If not advised otherwise, derive the mathematical solution for a problem from basic principles or general laws (Newton's laws, the Maxwell equations, the Schrödinger equation, etc.).
- You may use a calculator for basic operations only (i.e., not for referring to notes stored in memory, symbolic algebra, symbolic and numerical integration, etc.) The use of cell phones, tablets, and laptops is not permitted.
- Show your work and/or explain your reasoning in *all* problems, as the graders are not able to read minds. Even if your final answer is correct, not showing your work and reasoning will result in a *substantial* penalty.
- Write your work and reasoning in a neat, clear, and logical manner so that the grader can follow it. Lack of clarity is likely to result in a substantial penalty.

Problem 1: Classical Mechanics (CM 1)

A bead of mass m slides without friction on a circular hoop of radius R. The hoop rotates about a vertical axis through its center with constant angular speed ω. Gravity acts downward with acceleration g. The position of the bead is described by the angle θ measured from the vertical (see Figure 1).

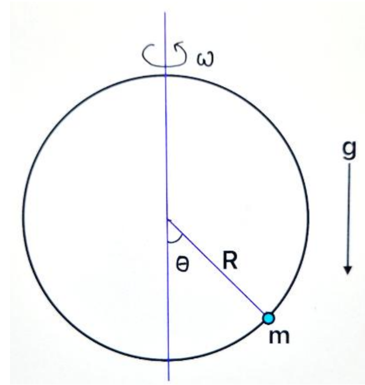
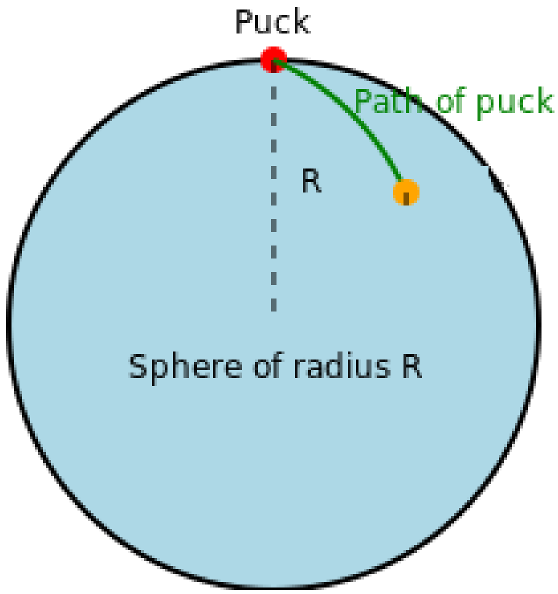


Figure 1

(1) Write the Lagrangian in terms of the generalized coordinate θ, and derive the equation of motion for θ.
(2) Show that two stable equilibrium positions are possible at $\theta = 0$ and $\cos\theta = g/(\omega^2 R)$.
(3) Compute the frequencies of small oscillations about each equilibrium position.
(4) Determine the critical angular speed ω_c above which the equilibrium at $\theta = 0$ becomes unstable, and a stable equilibrium appears at $\theta \neq 0$.

Problem 2: Classical Mechanics (CM 2)

A small frictionless puck is perched at the top of a fixed sphere of radius R. The puck is given a tiny nudge so that it begins to slide down. Through what vertical height will it descend before it leaves the surface of the sphere?



Problem 3: Electromagnetism (EM 1)

A planar wire loop, of negligible thickness, has a radius R and carries a counterclockwise current I. The loop is centered at the origin of the x-y plane, so that the z axis passes through the center of the loop.

(a) Using the Biot-Savart Law, find the magnetic field along the z-axis.

(b) Find the magnetic field in the x-y plane. Express your result in terms of a single integral on an angular variable ϕ.

(c) A small coin, of thickness d, radius a, and conductivity σ is placed at the center of, and coplanar with, the loop. If the current in the loop is $I(t) = I_0 \cos(\omega t)$, find the time-averaged power loss, assuming that $d \ll a \ll R$ and that the loop is a perfect conductor. You may assume that the frequency is low enough that retardation effects may be ignored.

Hint 1: Use your result from part (a) specialized to the point $z = 0$; i.e., find $B_z(0)$.

Hint 2: The Ohmic power density expression is $\frac{dP}{dV} = \boldsymbol{j} \cdot \boldsymbol{E}$, where V is the volume, $\boldsymbol{j}$ is the current density, and $\boldsymbol{E}$ is the electric field.

Problem 4: Electromagnetism (EM 2)

A solid dielectric spherical ball of radius R carries a spherically symmetric volume charge distribution of total charge Q. As a result of this symmetry, the electric field strength inside the ball ($r \leq R$) is $E(r) = E_{\text{max}}\, r^4/R^4$.

(a) What is E_{max} in terms of Q and R?

(b) Find an expression of the volume charge density $\rho(r)$ inside the ball as a function of r.

(c) Verify that your charge density gives the total charge Q when integrated over the volume of the ball.

Solution to Problem 1: Classical Mechanics (CM 1)

(1)

Kinetic Energy: $T = \frac{1}{2}m[R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta]$
Potential Energy: $V = -mgR\cos\theta$
Lagrangian: $\mathcal{L} = T - V = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\omega^2\sin^2\theta + mgR\cos\theta$

Equation of Motion:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

Compute:

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mR^2\dot{\theta}, \frac{d}{dt} = mR^2\ddot{\theta}$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta$$

So:

$$mR^2\ddot{\theta} - (mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta) = 0$$

$$\boxed{\ddot{\theta} = \sin\theta\left(\omega^2\cos\theta - \frac{g}{R}\right)}$$

(2) Equilibrium

Set $\ddot{\theta} = 0$:

$$\sin\theta\,(\omega^2\cos\theta - g/R) = 0$$

Solutions:

$$\sin\theta = 0 \implies \theta = 0 \text{ or } \pi$$

or

$$\cos\theta = \frac{g}{\omega^2 R}$$

Condition: $\omega^2 R \geq g$ for the second solution to exist.

(3) Frequency (Ω) of small oscillations

Case 1: Around $\boldsymbol{\theta_0 = 0}$

Expand:

$$\sin\theta \approx \theta, \cos\theta \approx 1$$

Equation:

$$\ddot{\theta} \approx \theta(\omega^2 - g/R)$$

So:

$$\boxed{\Omega^2 = g/R - \omega^2}, \qquad \Omega = \sqrt{g/R - \omega^2}$$

Stable if $g/R > \omega^2$.

Case 2: Around θ_0 with $\cos\ \theta_0 = g/(\omega^2 R)$

Let $\theta = \theta_0 + \eta$, linearize:

$$\ddot{\eta} \approx f'(\theta_0)\eta, \qquad f(\theta) = \sin\theta\,(\omega^2\cos\ \theta - g/R)$$

Compute derivative:

$$f'(\theta) = \cos\theta\,(\omega^2\cos\ \theta - g/R) + \sin\theta\,(-\omega^2\sin\ \theta)$$

At θ_0:

$$\omega^2\cos\ \theta_0 - g/R = 0 \implies f'(\theta_0) = -\omega^2\sin^2\theta_0$$

So:

$$\Omega^2 = \omega^2\sin^2\theta_0$$

But:

$$\sin^2\theta_0 = 1 - \cos^2\theta_0 = 1 - (g^2/(\omega^4 R^2))$$

Final:

$$\boxed{\Omega = \omega\sqrt{1 - \frac{g^2}{\omega^4 R^2}}}$$

(4) Critical angular speed

From stability at $\theta = 0$:

$$g/R > \omega^2 \implies \boxed{\omega_c = \sqrt{\frac{g}{R}}}$$

Above this, bottom equilibrium becomes unstable, and new stable equilibrium appears at $\theta \neq 0$.

Solution to Problem 2: Classical Mechanics (CM 2)

To simplify this problem, let θ be the angle from the puck's starting point to its current position on the sphere such that $\theta = 90$ degrees is when the puck is at the sphere's equator and let m be the mass of the puck.

Next, let us find speed of the puck in terms of angle traveled. The puck starts with a potential energy of mgR. After traversing an angle of θ, it will have a potential energy of $mgR\cos(\theta)$ and have a kinetic energy of $mv^2/2$. From conservation of energy, we have

$$mgR = mgR\cos\theta + \frac{1}{2}mv^2$$

We can then find the speed squared in terms of the angle as

$$v(\theta)^2 = 2gR(1 - cos\theta)$$

The puck will remain in contact with the sphere as long as the normal force is positive. It will lose contact immediately after the normal force is zero.

Using Newton's 2nd Law in the radial direction and noting we must be undergoing circular motion, we have

$$mg\cos\theta - N = \frac{mv^2}{R}$$

and equivalently, v^2 must satisfy

$$v^2 = gR\cos\theta - NR/m$$

to stay on the sphere.

The puck will then lose contact when $N = 0$. We can find when the happens by requiring θ to satisfy both the conservation of energy and Newton's law equations.

Then

$$2gR(1 - cos\theta) = gR\cos\theta \rightarrow \cos\theta = 2/3$$

Finally, we can figure out the vertical height. Let h be the height of above the center of the sphere and d be the vertical distance descended by the puck such that $h + d = R$.

We also know that $2/3 = \cos\theta = h/R$, so $h = (2/3)R$.

Therefore, the vertical distance descended must be $R/3$.

Solution to Problem 3: Electromagnetism (EM 1)

(a) Field on the z–axis (Biot–Savart). Choose source point on the loop

$$\mathbf{r}'(\phi) = R\cos\phi\,\hat{x} + R\sin\phi\,\hat{y}, \qquad d\boldsymbol{\ell} = \frac{d\mathbf{r}'}{d\phi}d\phi = R(-\sin\,\phi\,\hat{x} + \cos\,\phi\,\hat{y})\,d\phi.$$

Field point on axis: $\mathbf{r} = (0,0,z)$, so

$$\mathbf{s} = \mathbf{r} - \mathbf{r}' = (-R\cos\phi\,, -R\sin\phi\,, z), \qquad s^2 = R^2 + z^2.$$

Biot–Savart: $d\mathbf{B} = (\mu_0 I/4\pi)\,d\boldsymbol{\ell} \times \mathbf{s}/s^3$. Compute the cross product,

$$d\boldsymbol{\ell} \times \mathbf{s} = (Rz\cos\,\phi\,\hat{x} + Rz\sin\,\phi\,\hat{y} + R^2\,\hat{z})\,d\phi.$$

Integrating over $\phi \in [0,2\pi]$ leaves only the $\hat{z}$ component, so

$$B_z(0,0,z) = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}$$

At the center $z = 0$, $B_z(0) = \mu_0 I/(2R)$. (Sign consistent with right-hand rule.)

(b) Field in the plane (single integral). By rotational symmetry evaluate at $\mathbf{r} = (\rho, 0,0)$. Then

$$\mathbf{s} = (\rho - R\cos\phi\,, -R\sin\phi\,, 0), \qquad s^2 = \rho^2 + R^2 - 2\rho R\cos\,\phi.$$

Compute

$$d\boldsymbol{\ell} \times \mathbf{s} = (R^2 - R\rho\cos\,\phi)\,\hat{z}\,d\phi.$$

Hence the axial component at radius ρ is

$$B_z(\rho) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{R^2 - R\rho\cos\,\phi}{(\rho^2 + R^2 - 2\rho R\cos\,\phi)^{3/2}}\,d\phi$$

This integral can be expressed in terms of complete elliptic integrals with modulus $k^2 = \frac{4R\rho}{(R+\rho)^2}$:

$$B_z(\rho) = \frac{\mu_0 I}{2\pi(R+\rho)}[K(k) + \frac{R-\rho}{R+\rho}E(k)].$$

Limits: $\rho \to 0$ recovers $\mu_0 I/(2R)$; far field shows dipole $\propto \rho^{-3}$.

(c) Power dissipated in a small conducting coin.

A thin coin of radius a, thickness d and conductivity σ is centered, and coplanar with the loop.

Let $I(t) = I_0\cos\,(\omega t)$. Assume: quasi-static (no retardation), loop is an ideal current source, coin is small and thin, $d \ll a \ll R$, and skin depth $\delta_{\text{coin}} \gg d$.

Take the on-axis field at the coin from (a) (at the center): $B_z(t) \approx B_0 \cos\ \omega t$, $B_0 = \mu_0 I_0/(2R)$.

Flux through radius $r \leq a$: $\Phi(r,t) = \pi r^2 B_z(t)$. Faraday: $\oint E_\phi\, d\ell = -\dot{\Phi}$ gives

$$2\pi r E_\phi(r,t) = -\pi r^2 \dot{B}_z(t)$$

$$E_\phi(r,t) = \frac{r}{2}\omega B_0 \sin(\omega t).$$

Ohmic power density: $\frac{dP}{dV} = \boldsymbol{j} \cdot \boldsymbol{E} = \sigma E_\phi^2$. Time average $\langle \sin^2 \omega t \rangle = 1/2$, so

$$\langle \frac{dP}{dV} \rangle = \frac{\sigma \omega^2 B_0^2}{8} r^2.$$

Integrate over coin volume: $dV = 2\pi r\, dr\, dz$, $0 \leq r \leq a$, $0 \leq z \leq d$:

$$\int_0^d dz \int_0^a 2\pi r (\frac{\sigma \omega^2 B_0^2}{8} r^2)\, dr = \frac{\pi a^4 d\, \sigma \omega^2 B_0^2}{16}.$$

Therefore

$$\langle P \rangle = \frac{\pi a^4 d\, \sigma\, \omega^2 B_0^2}{16} = \frac{\mu_0^2 \pi a^4 d\, \sigma\, \omega^2 I_0^2}{64 R^2}$$

Solution to Problem 4: Electromagnetism (EM 2)

(a) This can be approached through Gauss's law. The field and charge are spherically symmetric, so the flux integral at some radius $r \le R$ is straightforward

$$\Phi_{\mathrm{E}} = \oint \vec{E} \cdot \mathrm{d}\vec{A} = \oint E(r)\, \mathrm{d}A = E(r) \oint \mathrm{d}A = E(r) \cdot (4\pi r^2)$$

Note that at $r = R$, $E(R) = E_{\max}$, so the flux integral becomes $\Phi_E = 4\pi R^2 E_{\max}$.
Take a gaussian surface to be a sphere of radius R, sharing its center with the ball of charge. Then the enclosed charge is simply the total charge, $q_{\mathrm{enc}} = Q$.
Applying Gauss's law gives

$$4\pi R^2 E_{\max} = \frac{Q}{\epsilon_0} \quad \Rightarrow \quad E_{\max} = \frac{Q}{4\pi\epsilon_0 R^2}$$

(b) We can write the electric field strength for $r \le R$ as

$$E(r) = \frac{Q}{4\pi\epsilon_0 R^6} r^4$$

For $r \le R$, let $q(r)$ represent the charge enclosed by a concentric gaussian surface or radius r. This charge can be found by integrating the volume charge density over the volume enclosed by the gaussian surface,

$$q(r) = \int \rho\, \mathrm{d}\tau = \int_0^r \rho(r') 4\pi (r')^2\, \mathrm{d}r'$$

Applying Gauss's law at $r \le R$, we have

$$4\pi r^2 \cdot E(r) = \frac{q(r)}{\epsilon_0} \quad \Rightarrow \quad q(r) = \frac{Q}{R^6} r^6$$

Setting the expressions for $q(r)$ equal gives us

$$\int_0^r \rho(r') 4\pi (r')^2\, \mathrm{d}r' = \frac{Q}{R^6} r^6$$

We then differentiate with respect to r to find $\rho(r)$

$$\rho(r) 4\pi r^2 = \frac{6Q}{R^6} r^5 \quad \Rightarrow \quad \rho(r) = \frac{3Q}{2\pi R^6} r^3$$

As a quick check, the dimensions of ρ are charge per volume, as expected.

(c) Now we can integrate $\rho(r)$ over the volume of the ball to verify the total charge.

$$q(R) = \int_0^R \rho(r) 4\pi r^2\, \mathrm{d}r = \int_0^R \left(\frac{3Q}{2\pi R^6} r^3\right)(4\pi r^2)\, \mathrm{d}r = \frac{Q}{R^6} \int_0^R 6r^5\, \mathrm{d}r = \frac{Q}{R^6} [r^6]_0^R = Q$$