

PHYSICAL REVIEW LETTERS

VOLUME 82

4 JANUARY 1999

NUMBER 1

Complete Control of Hamiltonian Quantum Systems: Engineering of Floquet Evolution

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(Received 30 April 1998)

We propose a method for controlling the unitary evolution of quantum systems by switching on and off alternatively two distinct constant perturbations. We show how to find appropriate switching times in order to attain any desired evolution. We suggest an experimental realization of our method in controlling the translational motion of cold atoms. [S0031-9007(98)08072-7]

PACS numbers: 03.65.-w, 03.75.-b, 32.80.Qk

One encounters the problem of quantum control in several fields of contemporary physics and chemistry, such as molecular dynamics in laser fields [1,2] and quantum optics [3–6]. A few examples of complete control of the quantum state by conditional measurements [7], by adiabatic transport [8], or by unitary evolution [9] have been already proposed for the particular quantum system of atoms interacting with quantized electromagnetic field in a single-mode resonator. But to what extent is it possible to control the quantum dynamics in the general case? In this Letter we show that one can obtain complete control not only over the quantum state but also over the unitary evolution of a generic Hamiltonian system. It can be achieved simply by switching on and off two distinct perturbations \hat{V}_A and \hat{V}_B in an alternating sequence. For an N -level system [10] the sequence is periodic, and each period consists of N^2 time intervals, t_1, t_2, \dots, t_{N^2} , which are found by solving the “inverse Floquet problem” [11] as described below. In other words, in order to control the system the perturbation \hat{V}_A should be applied during time t_1 followed by the perturbation \hat{V}_B during time t_2 , and then again \hat{V}_A during time t_3 followed by \hat{V}_B during time t_4 , and so forth, altogether N^2 times. After the last interval t_{N^2} the sequence repeats itself.

We emphasize the difference between the control of evolution and the control over a quantum state. The first means free choice of all elements of $N \times N$ evolution matrix \hat{U} , restricted only by the unitarity condition. Inversion or displacement of all wave functions or maintaining a quantum state intact in the course of time

without specifying this state are just three examples of many objectives that can be achieved only by this type of control. The second means free choice of N -component state vector $|f\rangle = \hat{U}|i\rangle$ corresponding to a given initial state $|i\rangle$, whereas the evolution of other states is ignored. It is an easier task that requires only N control parameters and can be performed in many different ways, including, in particular, the control of the evolution operator \hat{U} .

Our aim is to control the system, of an unperturbed Hamiltonian \hat{H}_0 , in such a way that after each period it will be effectively evolving according to an arbitrarily prescribed Hermitian Hamiltonian $\hat{H}_{\text{eff}} \neq \hat{H}_0$. The set of all possible Hermitian Hamiltonians \hat{H}_{eff} comprises a linear space of N^2 real dimensions, and therefore, in order to exert complete control over the system we need to have at least N^2 real control parameters at our disposal. The interaction intervals t_1, t_2, \dots, t_{N^2} are meant to serve as these parameters.

At the end of each period, at time $t = \sum_{n=1}^{N^2} t_n$, the evolution of the system is given by the unitary operator

$$\hat{U}(t) = e^{-i\hat{B}t_{N^2}} e^{-i\hat{A}t_{N^2-1}} \dots e^{-i\hat{B}t_2} e^{-i\hat{A}t_1}, \quad (1)$$

where we denote $\hat{A} = \hat{H}_0 + \hat{V}_A$; $\hat{B} = \hat{H}_0 + \hat{V}_B$ and set $\hbar = 1$. Hence, complete control of the evolution over a period means that for any given \hat{H}_{eff} one can find a sequence of positive times t_1, t_2, \dots, t_{N^2} such that

$$e^{-i\hat{B}t_{N^2}} e^{-i\hat{A}t_{N^2-1}} \dots e^{-i\hat{B}t_2} e^{-i\hat{A}t_1} = e^{-i\hat{H}_{\text{eff}}\tau} \quad (2)$$

holds for some effective evolution time τ .

In order to ensure complete control the operators \hat{A} and \hat{B} should apparently satisfy some conditions. We conjecture, in lack of a rigorous mathematical proof, that complete control is attainable if the set of commutators $[\hat{A}, [\hat{A}, \dots [\hat{A}, \hat{B}] \dots]]$ and $[\hat{B}, [\hat{B}, \dots [\hat{B}, \hat{A}] \dots]]$ of all orders up to N^2 spans the entire space of N -dimensional Hermitian Hamiltonians, that is, if these commutators form a complete basis for the $N \times N$ Hermitian matrices. This condition is implied by the requirements that (i) all eigenvalues and their pairwise differences are distinct both for \hat{A} and for \hat{B} , and (ii) that in the representation where \hat{A} is diagonal the matrix of \hat{B} has no zero elements and vice versa [12]. These conditions are met by all generic Hermitian pairs \hat{A} and \hat{B} . However, here we mainly focus on the case where all the commutators and, as a consequence, all the intervals t_n are of the same order of magnitude.

A straightforward attempt to solve numerically the system of N^2 nonlinear equations (2) for the times t_1, t_2, \dots, t_{N^2} results, even for relatively small systems, in cumbersome calculations which usually do not converge. Therefore, one needs a better numerical method that could rely on solving either a big but linear system of equations or a small system of nonlinear equations.

Our approach is based on the following idea: We should first find an ‘‘identity map’’ solution T_1, T_2, \dots, T_{N^2} for the particular case $\hat{H}_{\text{eff}} = 0$. If we succeed, the operator of the evolution over the period $T = \sum_{n=1}^{N^2} T_n$ is the identity transformation

$$\hat{U}(T) = e^{-i\hat{B}T_{N^2}} e^{-i\hat{A}T_{N^2-1}} \dots e^{-i\hat{B}T_2} e^{-i\hat{A}T_1} = 1. \quad (3)$$

For small variations $t_n = T_n + \delta t_n$ of the times T_n the expansion of the exponential factors in Eq. (1) yields the first order corrections

$$\begin{aligned} \hat{U}(t) &= e^{-i\hat{B}T_{N^2}} (1 - i\hat{B}\delta t_{N^2}) e^{-i\hat{A}T_{N^2-1}} (1 - i\hat{A}\delta t_{N^2-1}) \\ &\quad \times \dots e^{-i\hat{B}T_2} (1 - i\hat{B}\delta t_2) e^{-i\hat{A}T_1} (1 - i\hat{A}\delta t_1) \\ &\quad + o(\delta t_1, \delta t_2, \dots, \delta t_{N^2}) \\ &= 1 - i \sum_{n=1}^{N^2} \hat{H}_n \delta t_n + o(\delta t_1, \delta t_2, \dots, \delta t_{N^2}), \end{aligned} \quad (4)$$

with N^2 Hermitian operators \hat{H}_n

$$\begin{aligned} \hat{H}_1 &= e^{-i\hat{B}T_{N^2}} e^{-i\hat{A}T_{N^2-1}} \dots e^{-i\hat{B}T_2} e^{-i\hat{A}T_1} \hat{A}, \\ \hat{H}_2 &= e^{-i\hat{B}T_{N^2}} e^{-i\hat{A}T_{N^2-1}} \dots e^{-i\hat{B}T_2} \hat{B} e^{-i\hat{A}T_1}, \\ &\quad \dots \\ \hat{H}_{N^2} &= e^{-i\hat{B}T_{N^2}} \hat{B} e^{-i\hat{A}T_{N^2-1}} \dots e^{-i\hat{B}T_2} e^{-i\hat{A}T_1}. \end{aligned} \quad (5)$$

If, moreover, for our specific choice of timings T_n the operators \hat{H}_n turn out to be linearly independent, then by standard methods of linear algebra we can express any Hermitian operator \hat{H}_{eff} as a linear combination

$$\hat{H}_{\text{eff}} \tau = \sum_{n=1}^{N^2} \hat{H}_n \delta t_n. \quad (6)$$

Therefore we can rewrite Eq. (4) in the form

$$\hat{U}(t) = e^{-i\hat{H}_{\text{eff}}\tau} + o(\tau). \quad (7)$$

By taking τ small enough and by repeating the application of $\hat{U}(t)$ with the same timings $t_n = T_n + \delta t_n$ again and again we can emulate the evolution according to \hat{H}_{eff} over any finite duration of time and to any prescribed accuracy.

This scheme gives a solution of the control problem, although it may require so large a number of control periods that one cannot afford it in practice. Actually a better solution exists, which yields the desired control with utmost accuracy within a finite number of periods: For a relatively small but finite τ we can find variations δt_n for which Eq. (2) holds exactly. This can be achieved by an iterative process, where at each step we improve the accuracy of the corrections δt_n by linearizing Eq. (2) around a point t_1, t_2, \dots, t_{N^2} yet closer to the desired solution, starting with T_1, T_2, \dots, T_{N^2} as the initial point.

The problem of complete control is thus reduced to finding times T_n such that (a) $\hat{U}(\sum_{n=1}^{N^2} T_n) = 1$, and (b) the \hat{H}_n of Eq. (5) are linearly independent. Although we do not know of any general principle which would guarantee the existence of such timings, in practice, for all considered systems with N up to 16 we were able to find numerical values T_n by the following procedure. We first find a short sequence of N time intervals, T'_1, T'_2, \dots, T'_N , for which the unitary transformation

$$\hat{U}_r(T'_1, T'_2, \dots, T'_N) = e^{-i\hat{B}T'_N} \dots e^{-i\hat{B}T'_2} e^{-i\hat{A}T'_1} \quad (8)$$

is a ‘‘nondegenerate N th root of the unity operator’’; that is, it satisfies the condition

$$\hat{U}_r^N = 1 \quad (9)$$

and has distinct eigenvalues: $\lambda_q = e^{2\pi i q/N}$, $q = 1, 2, \dots, N$. Then we construct the long sequence of N^2 timings T_1, T_2, \dots, T_{N^2} by repeating N times the short sequence T'_1, T'_2, \dots, T'_N , and thus obtain timings T_n satisfying condition (a) by construction. Moreover, due to the nondegeneracy of \hat{U}_r we expect that they also satisfy condition (b), since in the generic case the nondegeneracy of \hat{U}_r ensures that all H_n are different and hence linearly independent [13].

We determine the required timings T'_1, T'_2, \dots, T'_N from the condition that the characteristic polynomial

$$\sum_{j=0}^N a_j(T'_1, \dots, T'_N) \lambda^j = \det[\lambda - \hat{U}_r(T'_1, \dots, T'_N)] \quad (10)$$

of the operator \hat{U}_r should have the form

$$\sum_{j=0}^N a_j(T'_1, T'_2, \dots, T'_N) \lambda^j = \lambda^N - 1, \quad (11)$$

which guarantees that its roots coincide with the eigenvalues λ_q . Equation (11) implies that all the coefficients a_j are zero, apart from a_N which is always equal to unity and

a_0 which is equal to $\det U_r$. Therefore, we can find values T'_1, T'_2, \dots, T'_N that satisfy it by minimizing the function

$$f(T'_1, T'_2, \dots, T'_N) = \sum_{j=0}^N |a_j(T'_1, T'_2, \dots, T'_N)|^2 \quad (12)$$

with respect to these N variables [14]. In this way we have found timings T'_1, T'_2, \dots, T'_N , for different systems of $N = 2, 4, \dots, 16$ levels, affording complete control over their evolution. We note that these solutions are not unique. Numerous solutions are scattered all over the space of N variables, and hence the choice of a specific one is open to optimization.

Two basic prerequisites ensure the applicability of the proposed method to a particular quantum system. First, the control time should be shorter than the shortest relaxation time $1/\Gamma$ of the system, which means $N_p \Gamma \sum_{n=1}^{N^2} T_n \ll 1$ where N_p is the number of control periods. Second, the phase error accumulated due to imperfections in the switching of the perturbations (for example, finite rising time and timing jitters) should be small. Assuming an uncertainty of δt in the switching timings and denoting by ω_{\max} the maximal frequency of the Hamiltonians \hat{A} and \hat{B} we arrive at the requirement $N_p N^2 \omega_{\max} \delta t \ll 1$.

Although the proposed method of control is applicable to any quantum object of a finite number of levels, the practical requirements of rapid and accurate switchings can be met easily only for a relatively slow system. The following example demonstrates the method in an experimentally realistic setting. Consider a cold atom confined in the presence of the gravitational field in a one dimensional potential pit of width L formed by two horizontal atomic mirrors [15], as shown in Fig. 1. The vertical motion of the atom along the z axis is described by the Hamiltonian

$$\hat{H}_0 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + mgz \quad (13)$$

and the boundary conditions $\psi(0) = \psi(L) = 0$ for the atomic wave function $\psi(z)$. Here m is the atomic mass and g is the gravitational acceleration.

To control the system we introduce two nonresonant standing electromagnetic waves, A and B , of different wavelengths, λ_A and λ_B , that interact with the atom via the induced-dipole-force Hamiltonians

$$\hat{V}_{A;B} = -\frac{\pi}{c} \alpha(\omega_{A;B}) I_{A;B} \sin^2(2\pi z/\lambda_{A;B}) \quad (14)$$

given by the product of the z -dependent field intensities $I_{A;B} \sin^2(2\pi z/\lambda_{A;B})$ and atomic susceptibilities $\alpha(\omega_{A;B})$ at the frequencies $\omega_{A;B} = 2c\pi/\lambda_{A;B}$ of the waves. For $\lambda_{A;B} \sim L$ the matrix elements of the perturbations (14) are significant for only the first few energy levels. For a rubidium atom of mass $m = 1.42 \times 10^{-22}$ g in a potential pit of $L = 2.5 \mu\text{m}$, and in the presence of

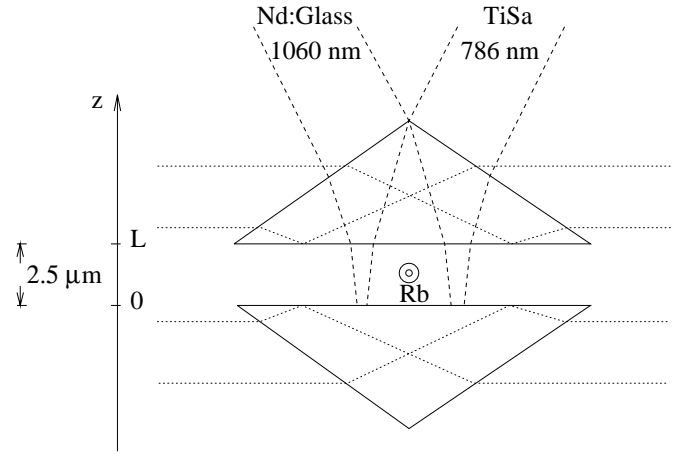


FIG. 1. Experimental setting for demonstrating the control method: Rubidium atom placed between two horizontal prisms forming two evanescent-wave atomic mirrors. Control of the vertical translational motion is achieved by nonresonant radiation of two lasers of different wavelengths that interact with the atom via the dipole force. To be specific we assume that both laser fields have nodes at the surface of the lower prism.

Nd-glass ($\lambda_A = 1060$ nm; $I_A \approx 22$ KW/cm²) and Ti-sapphire ($\lambda_B = 786$ nm; $I_B \approx 0.4$ KW/cm²) laser fields, we examine numerically the Hamiltonian (13) and perturbations (14) and find that only the first $N = 10$ levels are perturbed [16]. Therefore, we can restrict our attention to the problem of controlling just these levels.

In Fig. 2 we demonstrate two examples of control of the atom. With timings T_n of a few hundred microseconds given in Table I we arrive at an “identity map” evolution operator $\hat{U}(t) = \hat{U}_r^N = 1 \pm 10^{-15}$ that allows one to reconstruct completely any initial state after a single period [17]. In the presence of timing jitter $\delta t \sim 10^{-8}$ sec, typical of standard electrocommutation techniques, we restore the initial wave packet with accuracy 10^{-3} . For this system we also show the transformation of a narrow ($0.25 \mu\text{m}$) quasi-Gaussian wave packet into a two-hump distribution. It can be accomplished after $N_p = 15$ control periods with timings t_n , given in Table I, that are close to those of the “identity map.”

We conclude by summarizing the main results and by formulating several open questions. We have demonstrated that for any quantum system of finite number of states one can attain complete control of the Floquet evolution over a period by applying repeatedly an alternating sequence of two distinct time-independent perturbations. The control procedure relies on the solution of the “inverse Floquet problem” which specifies, for an N -level system, the lengths of N^2 time intervals per period during each of which one or the other of the two perturbations is applied. One finds this solution by linearizing the problem in the vicinity of the identity map solution, which gives the intervals for the identity transformation.

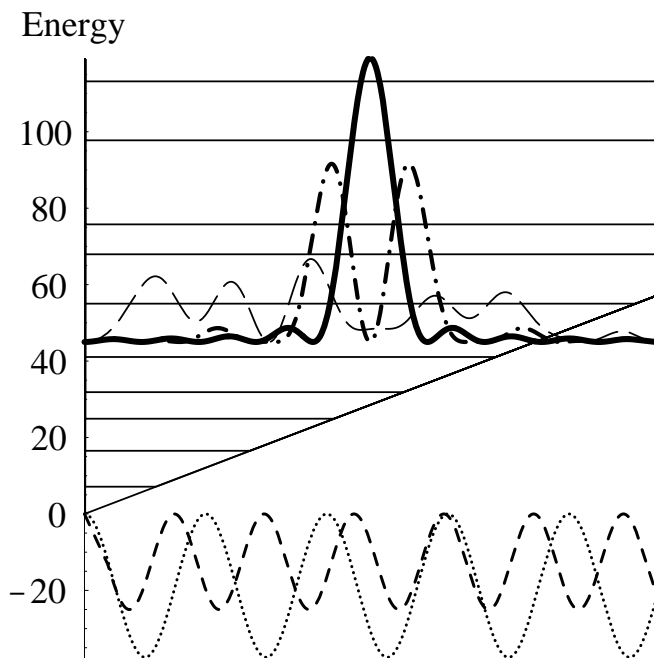


FIG. 2. Schematic view of the control of a trapped cold rubidium atom. Potential energy in units of $\pi^2 \hbar^2 / 2mL^2$ and first few energy levels E_i (solid lines) are shown along with the two control perturbations (shown not to scale) V_A (dotted line) and V_B (dashed line). The initial quasi-Gaussian probability distribution $|\psi(z)|^2 \sim \sum_{n=0}^5 \exp[-(2n/5)^2] \sin^2(2n\pi z/L)$ (bold line) is recovered accurately at time $T = \sum_{n=1}^{N^2} T_n$ after one control period with timings T_n given in Table I. The atomic distribution at the same time T for an uncontrolled atom (long-dashed line) differs from the initial distribution. The same initial distribution attains a desired two-hump shape after $N_p = 15$ control periods (bold dash-dotted line) with timings t_n given in Table I.

The determination of the identity map solution relies on the particular form $\lambda^N - 1$ of the characteristic polynomial of a nondegenerate N th root of the unity operator.

TABLE I. Identity map timings T_n for $N = 10$ levels of translation of a trapped cold atom, and timings $t_n = T_n + \delta t_n$ that allow one to split an initial quasi-Gaussian wave packet into a two-hump shaped distribution after 15 control periods as it is illustrated in Fig. 2. Only the first 10 out of the 100 periodic timings are presented. The rest of T_n repeat the values shown, and the rest of t_n are close to the values given here.

n	T_n (μsec)	t_n (μsec)	δt_n (μsec)
1	230.667	230.335	-0.332
2	333.121	330.169	-2.952
3	259.938	257.599	-2.339
4	248.654	247.880	-0.774
5	320.099	321.819	1.720
6	343.634	344.247	0.613
7	277.201	275.666	-1.535
8	235.567	239.881	4.314
9	331.637	330.457	-1.180
10	326.410	329.022	2.612

This nonlinear problem is solved numerically by minimizing the sum of absolute values of the coefficients of the characteristic polynomial of the evolution operator as a function of N time intervals. The solution is not unique and can be optimized. Although numerous solutions have been found for all considered quantum systems, the precise mathematical condition for their existence remains an open question. Another open question of practical importance is the possibility of generalizing the control method to the case of quantum systems with infinite number of states. In this context one can think of controlling a finite subsystem of an infinite system. Finally, the fact that for the controlled evolution all states of the system are accessible invites one to study the analogies with chaotic systems for which the sweep of all phase space is an intrinsic property.

G.H. thanks the Ministère Français des Affaires Etrangères for support.

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- [10] We assume N to be even without limiting of generality.
- [11] Evolution of quantum systems with periodic Hamiltonians is usually referred to as Floquet problem. See V.A. Yakubovich and V.M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients* (Wiley, New York, 1975).
- [12] We are grateful to Victor G. Kac for suggesting this condition. We also note that the complete control of quantum state (not evolution) can be performed in systems that do not satisfy this condition, as it is done in Ref. [9].
- [13] To illustrate this statement we note that for $\hat{U}(T) = \hat{U}_r^N = 1$ and for any $m \leq N$ and $0 \leq k' < k < N$ we

have (i) $\hat{H}_m = i\hat{U}_r^{-1} \frac{\partial \hat{U}_r}{\partial T_m}$ and (ii) $\hat{H}_{m+kN} = \hat{U}_r^{-k} \hat{H}_m \hat{U}_r^k \neq \hat{H}_{m+k'N}$. Conditions $\lambda_q = e^{2i\pi q/N}$ put only N constraints on N^2 parameters of the unitary matrix \hat{U}_r , and hence the first equation yields the N first \hat{H}_n linearly independent. In turn, the second equation ensures that all other \hat{H}_n are different, and hence in the generic case they are also linearly independent.

[14] Steepest descent minimization appears most suitable for this purpose; we just need to make sure that the local

minimum found is a global one (this is often the case). The timings obtained this way satisfy Eq. (9) up to an unimportant phase factor.

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[16] By the perturbation theory criterion $|V_{ij}| \ll |E_i - E_j|$.

[17] For the numbers actually shown in Table I (six digit accuracy) we obtain $\hat{U}(t) = 1 \pm 10^{-4}$.