# Mesoscopic phonon transmission through a nanowire-bulk contact

Chun-Min Chang and Michael R. Geller
*Department of Physics and Astronomy, University of Georgia, Athens, Georgia 30602-2451, USA*
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We calculate the frequency-dependent mesoscopic acoustic phonon transmission probability through the abrupt junction between a semi-infinite, one-dimensional cylindrical quantum wire and a three-dimensional bulk insulator, using a perturbative technique that is valid at low frequency. The system is described using elasticity theory, and traction-free boundary conditions are applied to all free surfaces. In the low-frequency limit the transmission probability vanishes as $\omega^2$, the transport being dominated by the longitudinal channel, which produces a monopole source of elastic radiation at the surface of the bulk solid. The thermal conductance between an equilibrated wire nonadiabatically coupled to a bulk insulator should therefore vanish with temperature as $T^3$.

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## I. INTRODUCTION

Electronic transport in a variety of mesoscopic systems has been successfully described by the theory of Landauer and Büttiker.[1–7] In this scattering approach, each fully propagating channel in a wire contributes $2\pi e^2/\hbar$ to the electrical conductance. Recently there have been experimental efforts to study phonons in the mesoscopic regime,[8] and beautiful experiments by Schwab *et al.*[9] and by Yung *et al.*[10] have observed the low-temperature universal thermal conductance quantum of $\pi k_B^2 T/6\hbar$ per vibrational channel. In these experiments, the nanowires were adiabatically (on the scale of the thermal wavelength) connected to thermalized phonon reservoirs,[11] and the observation of a thermal conductance varying linearly with temperature implies that very little phonon reflection occurred at the nanowire-bulk junctions.

In the presence of an abrupt, nonadiabiatic coupling between the nanowire and bulk reservoirs, phonons will scatter at the junctions and suppress the thermal conductance.[9,12–14] Cross and Lifshitz[13] have calculated the frequency-dependent acoustic phonon transmission probability $\mathsf{T}$ between a semi-infinite quantum wire of rectangular cross section and a thin plate with the same thickness as the wire, and find that $\mathsf{T} \propto \omega^{1/2}$ in the low-frequency limit. In Ref. 14 a short nanowire, modeled as a harmonic spring, abruptly connected at both ends to three-dimensional bulk insulators was found to have $\mathsf{T} \propto \omega^2$. These results suggest that such nanowires will eventually become thermal *insulators* at low temperatures, and the universal thermal conductance quantum will not be observed. However, such a crossover to insulating behavior has not yet been observed experimentally.

Understanding the scattering caused by nonadiabatic nanowire-bulk contacts will be essential for pushing phonon physics into the mesoscopic regime, as well as for the design and operation of thermal nanosensors such as calorimeters and bolometers. In this paper, we extend the previous work by calculating the mesoscopic acoustic phonon transmission probability through the abrupt junction between a semi-infinite, one-dimensional cylindrical quantum wire and a three-dimensional bulk insulator, using the low-frequency perturbative technique employed by Cross and Lifshitz.[13] The nanowire and bulk insulators are described using isotropic elasticity theory, and traction-free boundary conditions are applied to all free surfaces. In the low-frequency limit the transmission probability is found to vanish as $\omega^2$, with the low-frequency transport being dominated by the longitudinal channel.

In Sec. II we review the calculation of the long-wavelength vibrational modes of an infinitely long cylindrical elastic rod. The long wavelength limit of interest here is defined as $kb \ll 1$, where $b$ is the radius of the cylinder and $k$ is the wave number. In Sec. III we show that in the long-wavelength limit, the bulk solid produces a hard-wall boundary condition on the nanowire. In Sec. IV we calculate the displacement field in the three-dimensional bulk solid given an applied traction to its surface, using what is essentially an elastic Green's function method.[13,16] The frequency-dependent transmission probabilities for each of the four gapless modes are calculated in Sec. V, and our conclusions are given in Sec. VI.

## II. VIBRATIONAL MODES OF CYLINDRICAL WIRE

In this section, we will briefly review the elastic eigenmodes for an infinitely long cylindrical wave guide. In the long wavelength limit there are four branches, which include one torsional branch, one longitudinal branch, and two flexural branches.[15] We assign a numerical subscript to represent each branch, with "1" denoting the torsional branch, "2" denoting the longitudinal branch, and "3" and "4" denoting the flexural branches. These four branches are orthogonal. Also, cylindrical coordinates are used below.

We assume an isotropic elastic continuum with transverse and longitudinal sound speeds

$$c_t = \sqrt{\frac{\mu}{\rho}} \quad \text{and} \quad c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \tag{1}$$

where $\rho$ is the mass density, and $\lambda$ and $\mu$ are the Lamé constants.

### A. Branch 1: torsional

The displacement field is given by

$$\mathbf{u}_1(\mathbf{r},t) = r\, e^{i(kz-\omega_1 t)}\mathbf{e}_\theta, \tag{2}$$

with dispersion relation

 

$$\omega_1 = c_t k. \tag{3}$$

The stress tensor elements acting on a cross section of the rod are

$$\sigma_{1rz} = \mu\left(\frac{\partial u_{1r}}{\partial z} + \frac{\partial u_{1z}}{\partial r}\right) = 0,$$

$$\sigma_{1\theta z} = \mu\left(\frac{\partial u_{1\theta}}{\partial z} + \frac{1}{r}\frac{\partial u_{1z}}{\partial \theta}\right) = i\mu k r e^{i(kz-\omega_1 t)},$$

$$\sigma_{1zz} = \left[\lambda(\nabla\cdot\mathbf{u}_1) + 2\mu\frac{\partial u_{1z}}{\partial z}\right] = 0. \tag{4}$$

### B. Branch 2: longitudinal

The displacement field is

$$\mathbf{u}_2(\mathbf{r},t) = [f_r(r)\mathbf{e}_r + f_z(r)\mathbf{e}_z]e^{i(kz-\omega_2 t)}, \tag{5}$$

where

$$f_r(r) = -A_2\alpha J_1(\alpha r) + B_2 ikJ_1(\beta r),$$

$$f_z(r) = A_2 ikJ_0(\alpha r) - B_2\beta J_0(\beta r). \tag{6}$$

$J_n(r)$ is the Bessel function of $n$th order. $\alpha$ and $\beta$ are constants determined by the boundary conditions. Furthermore,

$$\alpha = \sqrt{\frac{\omega_2^2}{c_l^2} - k^2} \quad \text{and} \quad \beta = \sqrt{\frac{\omega_2^2}{c_t^2} - k^2}, \tag{7}$$

and

$$\frac{A_2}{B_2} = \frac{\beta^2 - k^2}{2i\alpha k}\frac{J_1(\beta b)}{J_1(\alpha b)}. \tag{8}$$

The relevant stresses are

$$\sigma_{2rz} = \mu\left[ikf_r(r) + \frac{df_z(r)}{dr}\right]e^{i(kz-\omega_2 t)},$$

$$\sigma_{2\theta z} = 0,$$

$$\sigma_{2zz} = \left[\lambda\left(\frac{df_r(r)}{dr} + \frac{f_r(r)}{r}\right) + ik(\lambda+2\mu)f_z(r)\right]e^{i(kz-\omega_2 t)}. \tag{9}$$

In the long wavelength $kb \ll 1$ limit, $\omega_2 = c_0 k$,

$$\alpha = ik\sqrt{1 - \frac{c_0^2}{c_l^2}} \quad \text{and} \quad \beta = k\sqrt{\frac{c_0^2}{c_t^2} - 1}, \tag{10}$$

where $c_0$ is related to Young's modulus of elasticity $E$ by

$$c_0 = \sqrt{\frac{E}{\rho}} = c_t\sqrt{\frac{3c_l^2 - 4c_t^2}{c_l^2 - c_t^2}}. \tag{11}$$

To leading order

$$\mathbf{u}_2(\mathbf{r},t) = \mathbf{e}_z\, e^{i(kz-\omega_2 t)}, \tag{12}$$

and the stresses are

$$\sigma_{2rz} = 0,$$

$$\sigma_{2\theta z} = 0,$$

$$\sigma_{2zz} = i\mu\frac{c_0^2}{c_t^2}k\, e^{i(kz-\omega_2 t)} = i\mu\left(\frac{3-4p^2}{1-p^2}\right)k\, e^{i(kz-\omega_2 t)}, \tag{13}$$

where $p \equiv c_t/c_l$ is the ratio of the transverse to longitudinal sound speed.

### C. Branch 3: $x$-polarized flexural

The displacement field is

$$\mathbf{u}_3(\mathbf{r},t) = [g_r(r)\cos\,\theta\mathbf{e}_r + g_\theta(r)\sin\,\theta\mathbf{e}_\theta + g_z(r)\cos\,\theta\mathbf{e}_z]e^{i(kz-\omega_3 t)}, \tag{14}$$

where

$$g_r(r) = \frac{dJ_1(\alpha r)}{dr} + B_3\frac{dJ_1(\beta r)}{dr} + C_3\frac{J_1(\beta r)}{r},$$

$$g_\theta(r) = -\frac{J_1(\alpha r)}{r} - B_3\frac{J_1(\beta r)}{r} - C_3\frac{dJ_1(\beta r)}{dr},$$

$$g_z(r) = ikJ_1(\alpha r) - iB_3\frac{\beta^2}{k}J_1(\beta r). \tag{15}$$

$B_3$ and $C_3$ are constants. The stresses are given by

$$\sigma_{3rz} = \mu\left(ikg_r(r) + \frac{dg_z(r)}{dr}\right)\cos\,\theta\, e^{i(kz-\omega_3 t)},$$

$$\sigma_{3\theta z} = \mu\left(ikg_\theta(r) - \frac{g_z(r)}{r}\right)\sin\,\theta\, e^{i(kz-\omega_3 t)},$$

$$\sigma_{3zz} = \left[\lambda\left(\frac{dg_r(r)}{dr} + \frac{g_r(r)+g_\theta(r)}{r}\right) + i(\lambda+2\mu)kg_z(r)\right]\cos\,\theta\, e^{i(kz-\omega_3 t)}. \tag{16}$$

In the $kb \ll 1$ limit,

$$\omega_3 = \frac{c_0}{2}bk^2, \tag{17}$$

$$\alpha = ik\sqrt{1 - \left(\frac{c_0 bk}{2c_l}\right)^2}, \tag{18}$$

and

$$\beta = ik\sqrt{1 - \left(\frac{c_0 bk}{2c_t}\right)^2}. \tag{19}$$

To leading order,

$$\begin{aligned}\mathbf{u}_3(\mathbf{r},t) &= (\cos\,\theta\mathbf{e}_r - \sin\,\theta\mathbf{e}_\theta + ikx\mathbf{e}_z)e^{i(kz-\omega_3 t)} \\ &= (\mathbf{e}_x - ikx\mathbf{e}_z)e^{i(kz-\omega_3 t)},\end{aligned} \tag{20}$$

and the stresses are

$$\sigma_{3xz} = \frac{i\mu}{4}\left[\left(1 + \frac{c_0^2}{c_t^2}\right)(b^2 - x^2) - \left(3 - \frac{c_0^2}{c_t^2}\right)y^2\right]k^3 e^{i(kz-\omega_3 t)},$$

$$\sigma_{3yz} = \frac{i\mu}{2}\left(1 - \frac{c_0^2}{c_t^2}\right)k^3 xy e^{i(kz-\omega_3 t)},$$

$$\sigma_{3zz} = \mu \frac{c_0^2}{c_t^2} k^2 x \; e^{i(kz-\omega_3 t)}. \tag{21}$$

The rod bends in the $xz$ plane.

### D. Branch 4: $y$-polarized flexural

An independent flexural mode can be found by letting the rod bend in the $y$ direction. In the long-wavelength limit,

$$\mathbf{u}_4(\mathbf{r},t) = (\mathbf{e}_y - iky\mathbf{e}_z)e^{i(kz-\omega_3 t)}, \tag{22}$$

and the stresses on the surface normal to $z$ are

$$\sigma_{4yz} = \frac{i\mu}{4}\left[\left(1 + \frac{c_0^2}{c_t^2}\right)(b^2 - y^2) - \left(\frac{c_0^2}{c_t^2} - 3\right)x^2\right]k^3 e^{i(kz-\omega_3 t)},$$

$$\sigma_{4xz} = -\frac{i\mu}{2}\left(\frac{c_0^2}{c_t^2} - 1\right)k^3 xy e^{i(kz-\omega_3 t)},$$

$$\sigma_{4zz} = \mu \frac{c_0^2}{c_t^2} k^2 y \; e^{i(kz-\omega_3 t)}. \tag{23}$$

## III. BOUNDARY CONDITIONS AT THE NANOWIRE-BULK INTERFACE

The essence of the perturbative method we use is as follows: In an abrupt junction geometry, the bulk solid presents a stiff boundary to the nanowire, so to zeroth order one calculates the vibrational modes of the wire assuming a zero-displacement boundary condition at the contact. The stress distributions associated with these vibrational modes is then calculated in the junction region. These zeroth-order vibrational modes, however, do not carry elastic energy because of the hard-wall boundary condition associated with the infinitely stiff bulk solid. Now one relaxes the hard-wall boundary condition, replacing it with the condition that elastic waves in the bulk are purely radiative, having outward moving components but no inward ones. The elastic energy radiated by the nanowire into the semi-infinite bulk solid is then computed using the actual elastic parameters of the bulk, and the ratio of incident to transmitted energy determines the transmission probability.

Thus, as explained, we will calculate the elastic stress on the surface of the three-dimensional bulk insulator, produced by the vibrating nanowire, by assuming that the bulk imposes a zero-displacement boundary condition on the nanowire. This is physically reasonable, and can be further justified by considering the bulk to be a thick wire with a radius $B$ much larger than the nanowire radius $b$.[13] Assuming a sound wavelength larger than both $b$ and $B$, the conservation of linear and angular momentum lead the zero-displacement boundary condition in the limit $B \gg b$.

We consider a semi-infinite cylindrical elastic nanowire of radius $b$, lying along the $z$ axis and connected at $z=0$ to a thicker cylinder of radius $B$. An incident elastic wave $\mathbf{u}_i = \mathbf{u}_i(r,\theta)e^{i(kz-\omega_3 t)}$ propagates from the nanowire to thick cylinder. $k$ is smaller than both $b$ and $B$ so both cylinders are still one dimensional, and $i=1,2,3,4$ denotes the branch. The scattering causes a reflected wave for $z<0$ and a transmitted wave for $z>0$. We can write the displacement field as

$$\mathbf{u}_i(r,\theta)e^{i(kz-\omega t)} + R_{ij}\mathbf{u}_j^*(r,\theta)e^{-i(kz+\omega t)} \quad \text{for } z < 0,$$

$$T_{ij}\mathbf{u}_j(r,\theta)e^{i(kz-\omega t)} \quad \text{for } z > 0. \tag{24}$$

Here $R_{ij}$ and $T_{ij}$ are the reflection and transmission coefficients, which are matrices in the channel indices.

The continuity of the displacement field, combined with the orthogonality of the vibrational eigenmodes, leads to

$$\delta_{ij} + R_{ij} = T_{ij}. \tag{25}$$

In the Appendix we show that in the $B \gg b$ limit, conservation of linear and angular momentum leads to

$$R_{ij} \to -\delta_{ij} \quad \text{and} \quad T_{ij} \to 0, \tag{26}$$

which means that elastic waves are flipped upon reflection, and no interbranch scattering occurs. This result is analogous to that obtained by Cross and Lifshitz in their thin-plate geometry.[13]

Linear combinations of the vibrational eigenfunctions of Sec. II can be used to satisfy the boundary conditions of Eq. (26). These displacement fields produce the following stress distributions at the $z=0$ interface: For the torsional mode we obtain

$$\sigma_{\theta z} = \begin{cases} 2i\mu k r^2 e^{-i\omega t} & \text{for } r < b, \\ 0 & \text{for } r > b. \end{cases} \tag{27}$$

For the longitudinal mode,

$$\sigma_{zz} = \begin{cases} 2i\mu \dfrac{c_0^2}{c_t^2} k e^{-i\omega t} & \text{for } r < b, \\ 0 & \text{for } r > b. \end{cases} \tag{28}$$

And for the $x$-polarized flexural mode, we find

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \quad \text{for } r > b, \tag{29}$$

$$\sigma_{xz} = \frac{i\mu}{2}\left[\left(1 + \frac{c_0^2}{c_t^2}\right)(b^2 - x^2) - \left(3 - \frac{c_0^2}{c_t^2}\right)y^2\right]k^3 e^{-i\omega_3 t},$$

$$\sigma_{yz} = i\mu\left(1 - \frac{c_0^2}{c_t^2}\right)k^3 xy e^{-i\omega_3 t},$$

$$\sigma_{zz} = 0, \tag{30}$$

for $r<b$. The stress distribution from the $y$-polarized flexural mode has the same form as (29) and (30) after exchanging $x$ and $y$.

### IV. 3D ELASTIC RESPONSE FUNCTION

Next we calculate the displacement field in the three-dimensional solid given the applied stress of Sec. III to its surface at $z=0$. For $r<b$, this is the stress distribution produced by the nanowire, and for $r>b$ it is the stress imposed by the traction-free boundary condition. The method we use here, which is essentially a Green's function method, is well known in elasticity theory.[13,16]

To find the displacement field in the bulk solid given the boundary conditions described above, a scalar potential $\phi$ and a vector potential $\mathbf{H}$ are introduced according to

$$\mathbf{u} = \nabla \phi + \nabla \times \mathbf{H}. \tag{31}$$

The wave equations for the potential fields are

$$\left(\frac{\partial^2}{\partial t^2} - c_l^2\nabla^2\right)\phi = 0, \quad \left(\frac{\partial^2}{\partial t^2} - c_t^2\nabla^2\right)\mathbf{H} = 0. \tag{32}$$

They can be written as

$$\phi(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_1 dk_2 f(k_1,k_2) e^{-i(k_1x+k_2y)} e^{ik_{l3}z} \tag{33}$$

and

$$\mathbf{H}(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_1 dk_2 \mathbf{h}(k_1,k_2) e^{-i(k_1x+k_2y)} e^{ik_{t3}z}, \tag{34}$$

where

$$k_{l3} = \sqrt{\frac{\omega^2}{c_l^2} - k_1^2 - k_2^2} \tag{35}$$

and

$$k_{t3} = \sqrt{\frac{\omega^2}{c_t^2} - k_1^2 - k_2^2}. \tag{36}$$

Here $f$ and $\mathbf{h}$ are the Fourier transforms of the potential fields $\phi$ and $\mathbf{H}$ at $z=0$.

Now, we can use Eq. (31) and choose the transverse "gauge" $\nabla\cdot\mathbf{H}=0$ to express the components of the displacement vector by the inverse Fourier transform $\mathcal{F}^{-1}$,

$$u_x(x,y,z) = -i\mathcal{F}^{-1}[k_1 f(k_1,k_2)e^{ik_{l3}z} + g_x(k_1,k_2)e^{ik_{t3}z}],$$

$$u_y(x,y,z) = -i\mathcal{F}^{-1}[k_2 f(k_1,k_2)e^{ik_{l3}z} + g_y(k_1,k_2)e^{ik_{t3}z}],$$

$$u_z(x,y,z) = i\mathcal{F}^{-1}\Bigg(k_{l3}f(k_1,k_2)e^{ik_{l3}z} - \frac{k_1 g_x(k_1,k_2) + k_2 g_y(k_1,k_2)}{k_{t3}}e^{ik_{t3}z}\Bigg), \tag{37}$$

where

$$g_x(k_1,k_2) = k_2 h_z(k_1,k_2) + k_{t3}h_y(k_1,k_2), \tag{38}$$

$$g_y(k_1,k_2) = -k_1 h_z(k_1,k_2) - k_{t3}h_x(k_1,k_2). \tag{39}$$

The stress at the boundary $z=0$ can also be expressed in terms of the inverse Fourier transforms, as

$$\sigma_{xz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)_{z=0} = \mu\mathcal{F}^{-1}\Bigg(2k_1k_{l3}f(k_1,k_2) + \frac{(k_{t3}^2 - k_1^2)g_x(k_1,k_2) - k_1k_2g_y(k_1,k_2)}{k_{t3}}\Bigg), \tag{40}$$

$$\sigma_{yz} = \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)_{z=0} = \mu\mathcal{F}^{-1}\Bigg(2k_2k_{l3}f(k_1,k_2) + \frac{(k_{t3}^2 - k_2^2)g_y(k_1,k_2) - k_1k_2g_x(k_1,k_2)}{k_{t3}}\Bigg), \tag{41}$$

$$\sigma_{zz} = \left(\lambda(\nabla\cdot\vec{u}) + 2\mu\frac{\partial u_z}{\partial z}\right)_{z=0} = \mu\mathcal{F}^{-1}[(k_1^2 + k_2^2 - k_{t3}^2)f(k_1,k_2) + 2k_1g_x(k_1,k_2) + 2k_2g_y(k_1,k_2)]. \tag{42}$$

By giving the boundary values of $\sigma_{xz}$, $\sigma_{yz}$, and $\sigma_{zz}$, we can find $f$, $g_x$, and $g_y$ from the equations above. If at least one of these three stresses is nonzero, we obtain

$$f = \frac{1}{\eta_0(k_1,k_2)}\Bigg[(k_1^2 + k_2^2 - k_{t3}^2)\mathcal{F}\left(\frac{\sigma_{zz}}{\mu}\right) + 2k_{t3}k_1\mathcal{F}\left(\frac{\sigma_{xz}}{\mu}\right) + 2k_{t3}k_2\mathcal{F}\left(\frac{\sigma_{yz}}{\mu}\right)\Bigg], \tag{43}$$

$$g_x = \frac{1}{\eta_0(k_1,k_2)}\Bigg[2k_{l3}k_{t3}k_1\mathcal{F}\left(\frac{\sigma_{zz}}{\mu}\right) + \left(\frac{\eta_1(k_1,k_2)}{k_{t3}} - 2k_{t3}k_1^2\right)\times\mathcal{F}\left(\frac{\sigma_{xz}}{\mu}\right) - \left(\frac{\eta_2(k_1,k_2)}{k_{t3}} + 2k_{t3}k_1k_2\right)\mathcal{F}\left(\frac{\sigma_{yz}}{\mu}\right)\Bigg], \tag{44}$$

and

$$g_y = \frac{1}{\eta_0(k_1,k_2)}\Bigg[2k_{l3}k_{t3}k_2\mathcal{F}\left(\frac{\sigma_{zz}}{\mu}\right) - \left(\frac{\eta_2(k_2,k_1)}{k_{t3}} + 2k_{t3}k_1k_2\right)\times\mathcal{F}\left(\frac{\sigma_{xz}}{\mu}\right) + \left(\frac{\eta_1(k_2,k_1)}{k_{t3}} - 2k_{t3}k_2^2\right)\mathcal{F}\left(\frac{\sigma_{yz}}{\mu}\right)\Bigg]. \tag{45}$$

Here

$$\eta_1(k_1,k_2) \equiv k_2^4 + k_1^2k_2^2 + k_1^2k_{t3}^2 + 2k_{t3}k_2^2(2k_{l3} - k_{t3}) + k_{t3}^4, \tag{46}$$

$$\eta_2(k_1,k_2) \equiv k_1k_2[k_1^2 + k_2^2 + k_{t3}(4k_{l3} - 3k_{t3})], \tag{47}$$

$$\eta_0(k_1,k_2) \equiv \eta_1(k_1,k_2) + \frac{k_1}{k_2}\eta_2(k_1,k_2). \tag{48}$$

Therefore, from Eq. (37), we can find the displacement vector at $z=0$ in terms of the boundary stresses,

$$u_x|_{z=0} = \mathcal{F}^{-1}\Bigg\{\frac{-i}{k_{t3}\eta_0(k_1,k_2)}\Bigg[k_1k_{t3}\eta_3(k_1,k_2)\mathcal{F}\left(\frac{\sigma_{zz}}{\mu}\right) + \eta_1(k_1,k_2)\mathcal{F}\left(\frac{\sigma_{xz}}{\mu}\right) - \eta_2(k_1,k_2)\mathcal{F}\left(\frac{\sigma_{yz}}{\mu}\right)\Bigg]\Bigg\}, \tag{49}$$

$$u_y|_{z=0} = \mathcal{F}^{-1}\left\{ \frac{-i}{k_{t3}\eta_0(k_1,k_2)} \left[ k_2 k_{t3} \eta_3(k_1,k_2) \mathcal{F}\left(\frac{\sigma_{zz}}{\mu}\right) + \eta_1(k_2,k_1)\mathcal{F}\left(\frac{\sigma_{yz}}{\mu}\right) - \eta_2(k_2,k_1)\mathcal{F}\left(\frac{\sigma_{xz}}{\mu}\right)\right]\right\}, \tag{50}$$

$$u_z|_{z=0} = \mathcal{F}^{-1}\left( \frac{-i}{\eta_0(k_1,k_2)} \left\{ \frac{\omega^2}{c_t^2} k_{l3} \mathcal{F}\left(\frac{\sigma_{zz}}{\mu}\right) - \eta_3(k_1,k_2) \times \left[ k_1 \mathcal{F}\left(\frac{\sigma_{xz}}{\mu}\right) + k_2 \mathcal{F}\left(\frac{\sigma_{yz}}{\mu}\right)\right]\right\}\right), \tag{51}$$

with

$$\eta_3(k_1,k_2) \equiv k_1^{\,2} + k_2^{\,2} + k_{t3}(2k_{l3} - k_{t3}). \tag{52}$$

## V. ENERGY TRANSMISSION FROM NANOWIRE TO BULK

Now we are ready to calculate the transmission probability, defined as the ratio of transmitted to incident elastic energy flux, for each of the four gapless branches. The energy current $I$ can be expressed as[17]

$$I = \left\langle \int_s dxdy \left( \frac{\partial u_x}{\partial t}\sigma_{xz} + \frac{\partial u_y}{\partial t}\sigma_{yz} + \frac{\partial u_z}{\partial t}\sigma_{zz}\right)\right\rangle_{z=0}, \tag{53}$$

where $\langle \cdots \rangle$ represents a time average and $\int_s dxdy$ is the surface integral over the $z=0$ cross section of the wire. In conventional complex notation for waves, this becomes

$$I = -\frac{\omega}{2}\mathrm{Im}\int_s dxdy(u_x\sigma_{xz}^* + u_y\sigma_{yz}^* + u_z\sigma_{zz}^*)_{z=0}. \tag{54}$$

We will calculate the energy current for the different branches separately.

### A. Torsional branch

The stress distribution (27) in rectangular coordinates is, for $\sqrt{x^2+y^2}<b$,

$$\sigma_{zz} = 0,$$
$$\sigma_{xz} = -2i\mu k y e^{-i\omega t},$$
$$\sigma_{yz} = 2i\mu k x e^{-i\omega t}, \tag{55}$$

and zero for $\sqrt{x^2+y^2} \geqslant b$. Using Eqs. (49)–(51) we obtain

$$I = -\frac{\omega}{2}\mathrm{Im}\int_s dxdy(u_x\sigma_{xz}^* + u_y\sigma_{yz}^*)_{z=0}$$
$$= \frac{\omega}{4\pi}\mathrm{Re}\int \frac{dk_1dk_2}{k_{t3}\eta_0(k_1,k_2)}\int_s dxdy e^{-i(k_1x+k_2y)} \times \left\{ \sigma_{xz}^*\left[\eta_1(k_1,k_2)\mathcal{F}\left(\frac{\sigma_{xz}}{\mu}\right) - \eta_2(k_1,k_2)\mathcal{F}\left(\frac{\sigma_{yz}}{\mu}\right)\right] + \sigma_{yz}^*\left[\eta_1(k_2,k_1)\mathcal{F}\left(\frac{\sigma_{yz}}{\mu}\right) - \eta_2(k_2,k_1)\mathcal{F}\left(\frac{\sigma_{xz}}{\mu}\right)\right]\right\} \tag{56}$$

$$= 2\mu k^2\omega\ \mathrm{Re}\int \frac{dk_1dk_2}{k_{t3}\eta_0(k_1,k_2)}\{\mathcal{F}[y]^*(\eta_1(k_1,k_2)\mathcal{F}[y] + \eta_2(k_1,k_2)\mathcal{F}[x]) + \mathcal{F}[x]^*(\eta_1(k_2,k_1)\mathcal{F}[x] + \eta_2(k_2,k_1)\mathcal{F}[y])\}. \tag{57}$$

Expanding the Fourier transforms $\mathcal{F}[x]$ and $\mathcal{F}[y]$ for small $kb$, and keeping only the leading terms,

$$\frac{1}{2\pi}\int_s x e^{-i(k_1x+k_2y)}dxdy \simeq \frac{-ik_1b^4}{8}, \tag{58}$$

$$\frac{1}{2\pi}\int_s y e^{-i(k_1x+k_2y)}dxdy \simeq \frac{-ik_2b^4}{8}, \tag{59}$$

we have

$$I = \frac{\mu b^8k^2\omega}{32}\mathrm{Re}\int \frac{dk_1dk_2}{k_{t3}\eta_0(k_1,k_2)}\left[k_2^{\,2}\left(\eta_1(k_1,k_2) + \frac{k_1}{k_2}\eta_2(k_1,k_2)\right) + k_1^{\,2}\left(\eta_1(k_2,k_1) + \frac{k_2}{k_1}\eta_2(k_2,k_1)\right)\right] \tag{60}$$

$$= \frac{\mu b^8k^2\omega}{32}\mathrm{Re}\int dk_1dk_2\frac{k_2^{\,2}+k_1^{\,2}}{k_{t3}} \tag{61}$$

$$= \frac{\pi}{24}\mu b^8k^5\omega. \tag{62}$$

By normalizing $I$ with the energy current $(\pi/4)\mu b^4 k\omega$ of the incident torsional wave, we obtain the transmission probability

$$\mathcal{T} = \tfrac{1}{6}b^4k^4. \tag{63}$$

### B. Longitudinal branch

Using Eq. (28),

$$I = -\frac{\omega}{2}\mathrm{Im}\int_0^b rdr\int_0^{2\pi}d\theta(u_z\sigma_{zz}^*)_{z=0}. \tag{64}$$

Then to leading order in $kb$,

$$I = \frac{\omega}{8\pi^2\mu}\left|\int_0^b rdr\int_0^{2\pi}d\theta\sigma_{zz}\right|^2\mathrm{Re}\int dk_1dk_2\frac{\omega^2k_{l3}}{c_t\eta_0(k_1,k_2)}$$
$$= \frac{\mu}{2c_t}\left(\frac{c_0^{\,2}}{c_t^{\,2}}b^2k\omega\right)^2\mathrm{Re}\int dk_1dk_2\frac{\omega k_{l3}}{c_t\eta_0(k_1,k_2)}. \tag{65}$$

Normalizing by the power $(\pi/2)\mu(c_0^{\,2}/c_t^{\,2})b^2k\omega$ carried by the incident wave leads to

$$\mathcal{T}(k) = \frac{c_0^{\,2}}{\pi c_t^3}b^2k\omega\ \mathrm{Re}\int dk_1dk_2\frac{\omega k_{l3}}{c_t\eta_0(k_1,k_2)} \tag{66}$$

$$= t_l b^2k^2, \tag{67}$$

where

$$t_l \equiv \frac{c_0^3}{\pi c_t^3}\mathrm{Re}\int dk_1 dk_2 \frac{\omega k_{l3}}{c_t \eta_0(k_1,k_2)}. \tag{68}$$

Assuming the materials to be made of Si, we have $p = 0.694$ and

$$\mathcal{T} = 1.91 b^2 k^2 \tag{69}$$

$$= 0.923\left(\frac{b\omega}{c_t}\right)^2. \tag{70}$$

### C. Flexural branches

By using flexural stress distribution (30), we find that to leading order in $kb$ the energy current is

$$\begin{aligned} I &= \frac{\omega}{2}\mathrm{Re}\int_s dxdy \mathcal{F}^{-1}\left[\frac{\eta_1(k_1,k_2)}{k_{t3}\eta_0(k_1,k_2)}\mathcal{F}\left(\frac{\sigma_{xz}}{\mu}\right)\right]\sigma_{xz}^* \\ &= \frac{\omega}{8\pi^2\mu}\left|\int_s \sigma_{xz} dxdy\right|^2 \mathrm{Re}\int dk_1 dk_2 \frac{\eta_1(k_1,k_2)}{k_{t3}\eta_0(k_1,k_2)} \\ &= \frac{\mu\omega k^6}{32\pi^2}\left[\int_s dxdy\left(1 + \frac{c_0{}^2}{c_t{}^2}\right)(b^2 - x^2) \right. \\ &\quad \left. + \left(\frac{c_0{}^2}{c_t{}^2} - 3\right)y^2\right]^2 \mathrm{Re}\int dk_1 dk_2 \frac{\eta_1(k_1,k_2)}{k_{t3}\eta_0(k_1,k_2)} \\ &= \frac{\mu}{32}\left(\frac{c_0{}^2}{c_t{}^2} b^4 k^3\right)^2 \omega\ \mathrm{Re}\int dk_1 dk_2 \frac{\eta_1(k_1,k_2)}{k_{t3}\eta_0(k_1,k_2)}. \end{aligned} \tag{71}$$

Normalizing by the energy current of the incident wave

$$I_{\mathrm{in}} = -\frac{\omega}{2}\mathrm{Im}\int_s dxdy(\sigma_{xz}^* - ikx\sigma_{zz}^*)_{z=0} \tag{72}$$

$$= \frac{\pi\mu}{4}\frac{c_0{}^2}{c_t{}^2} b^4 k^3 \omega, \tag{73}$$

where we have used Eqs. (20) and (21), the transmission probability is found to be

$$\mathcal{T} = \frac{1}{8\pi}\frac{c_0{}^2}{c_t{}^3} b^4 k^3 \omega\ \mathrm{Re}\int dk_1 dk_2 \frac{c_t \eta_1(k_1,k_2)}{\omega k_{t3}\eta_0(k_1,k_2)} \tag{74}$$

$$= t_f b^5 k^5 \tag{75}$$

where

$$t_f \equiv \frac{c_0{}^2}{16\pi c_t{}^3}\mathrm{Re}\int dk_1 dk_2 \frac{c_t \eta_1(k_1,k_2)}{\omega k_{t3}\eta_0(k_1,k_2)}. \tag{76}$$

Using $p = 0.694$ for Si, the transmission probability becomes

$$\mathcal{T} = 0.268\ b^5 k^5 \tag{77}$$

$$= 0.609\left(\frac{b\omega}{c_t}\right)^{5/2}. \tag{78}$$

Because of the cylindrical symmetry, the $y$-polarized flexural branch has the same transmission probability as the $x$-polarized branch.

## VI. CONCLUSIONS

On the left side of Table I we summarize the transmission probability results calculated above, as well as the low-frequency dispersion relations of the four gapless modes. For comparison with the results of Cross and Lifshitz[13] for a rectangular wire connected to a thin plate, we reproduce their results on the right-hand side of this table. In each case there are four gapless acoustic modes: one torsional, one longitudinal (or compressional) and two flexural bending modes. Also, the form of the dispersion relations are the same for both wires. For all branches the transmission probability to a three-dimensional bulk solid has a higher-order frequency dependence. This is at least partially a consequence of the higher vibrational density of states in the three-dimensional system as compared to a plate: For the longitudinal and $x$-polarized flexural branches, $\mathcal{T}$ is one order higher in $\omega$, consistent with the density of states enhancement.[18]

The phonon transmission probabilities can be used to calculate the mesoscopic thermal conductance between an equilibrated wire and bulk. According to the thermal Landauer formula,[12,17,19] a *total* transmission probability $\mathsf{T}(\omega)$ varying at low frequency as $\omega^\gamma$ will lead to a low-temperature thermal conductance varying with temperature as $G_{\mathrm{th}} \propto T^{\gamma+1}$. In our case, $\mathsf{T}(\omega)$ is a sum of the $\mathcal{T}$ for each channel. The thermal conductance between an equilibrated cylindrical wire nonadiabatically coupled to a bulk solid should therefore vanish with temperature as $T^3$.

Finally we comment on the applicability of our results to nanoscale phonon experiments, which do not consider infinitely long wires and perfectly sharp corners. For our theory to be valid, the wire must be longer than the sound wavelength, and the characteristic radii of curvature at the junction must be much smaller than this wavelength. Therefore, because of the first condition, our results will become invalid in the extreme low-temperature limit, and the conductance will cross over from our predicted $T^3$ scaling to some other behavior.

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## APPENDIX: MOMENTUM CONSERVATION AND HARD-WALL BOUNDARY CONDITION

Here we use linear and angular momentum conservation to derive Eq. (26) in the $B \gg b$ limit. First we equate the

TABLE I. (Left) Dispersion relations $\omega(k)$ of the low-frequency vibrational modes in a cylindrical nanowire, and transmission probabilities $\mathcal{T}$ through the junction with a three-dimensional bulk insulator, as a function of both $k$ and $\omega$. $t_l$ and $t_f$ are constants defined in Eqs. (68) and (76). In the low-frequency limit the total transmission probability vanishes as $\omega^2$, the transport being dominated by the longitudinal channel. (Right) Same quantities for a rectangular wire connected to a thin plate, reproduced from Ref. 13. Here $I_1$ and $I_2$ are Poisson-ratio-dependent numbers.

| Cylindrical nanowire (radius $b$) → semi-infinite space (3D solid) | | | | Rectangular nanowire (width $b$, thickness $d$) → thin plate (2D plate of thickness $d$) | | | |
|---|---|---|---|---|---|---|---|
| Branch | $\omega(k)$ | $\mathcal{T}(k)$ | $\mathcal{T}(\omega)$ | Branch | $\omega(k)$ | $\mathcal{T}(k)$ | $\mathcal{T}(\omega)$ |
| Torsional | $c_t k$ | $\frac{1}{6}(bk)^4$ | $\frac{1}{6}\left(\frac{b\omega}{c_t}\right)^4$ | Torsional | $\frac{2d}{b}c_t k$ | $I_2 bk$ | $I_2\frac{b}{2d}\frac{b\omega}{c_t}$ |
| Longitudinal | $c_0 k$ | $t_l(bk)^2$ | $t_l\left(\frac{b\omega}{c_0}\right)^2$ | Compressional | $c_0 k$ | $4bk$ | $4\frac{b\omega}{c_0}$ |
| Flexural ($x$-direction bending) | $\frac{1}{2}c_0 bk^2$ | $t_f(bk)^5$ | $4\sqrt{2}t_f\left(\frac{b\omega}{c_0}\right)^{5/2}$ | In-plane bending | $\frac{\sqrt{3}}{6}c_0 bk^2$ | $\frac{1}{3}(bk)^3$ | $2\sqrt[4]{\frac{4}{3}}\left(\frac{b\omega}{c_0}\right)^{3/2}$ |
| Flexural ($y$-direction bending) | same as $x$ direction | | | Normal-plane bending | $\frac{\sqrt{3}}{6}c_0 dk^2$ | $I_1 bk$ | $I_1\sqrt[4]{12}\left(\frac{b^2\omega}{dc_0}\right)^{1/2}$ |

torques produced by the thin and thick wires on each other. From the rotational stress

$$\sigma_{\theta z} = \mu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \tag{A1}$$

$$= \begin{cases} \sigma_{i\theta z}(r,\theta)e^{i(kz-\omega t)} + R_{ij}\sigma^*_{j\theta z}(r,\theta)e^{-i(kz+\omega t)} & \text{for } z < 0, \\ T_{ij}\sigma_{j\theta z}(r,\theta)e^{i(kz-\omega t)} & \text{for } z > 0, \end{cases} \tag{A2}$$

the torsional torque $\tau = \int_s r dr d\theta r \sigma_{\theta z}$ is found to be

$$\tau = \begin{cases} \dfrac{i\pi}{2}b^4(\delta_{i1}e^{ikz} - R_{i1}e^{-ikz})\mu k e^{-i\omega t} & \text{for } z = 0^-, \\ \dfrac{i\pi}{2}B^4 T_{i1}\mu k e^{i(kz-\omega t)} & \text{for } z = 0^+. \end{cases} \tag{A3}$$

Only torsional mode contributes to the torque.

By equating these torques we have

$$b^4(\delta_{i1} - R_{i1}) = B^4 T_{i1}. \tag{A4}$$

Combining this result with Eq. (25) leads to

$$R_{i1} = T_{i1} = 0, \quad i \neq 1, \tag{A5}$$

$$R_{11} = -\frac{B^4 - b^4}{B^4 + b^4}, \quad T_{11} = \frac{2b^4}{B^4 + b^4}. \tag{A6}$$

Taking the $B/b \to \infty$ limit then leads to $R_{11} \to -1$ and $T_{11} \to 0$.

An analogous result for $R_{i2}$ and $T_{i2}$ can be derived by considering the total force in $z$ direction,

$$F_z = \int r dr d\theta \sigma_{zz} \tag{A7}$$

$$= \begin{cases} i\pi b^2(\delta_{i2}e^{ikz} - R_{i2}e^{-ikz})\dfrac{c_0{}^2}{c_t{}^2}\mu k e^{-i\omega t} & \text{for } z < 0, \\ i\pi B^2 T_{ij}\dfrac{c_0{}^2}{c_t{}^2}\mu k e^{i(kz-\omega t)} & \text{for } z > 0, \end{cases} \tag{A8}$$

which gives

$$b^2(\delta_{i2} - R_{i2}) = B^2 T_{i2}, \tag{A9}$$

since only the longitudinal mode has a nonzero $F_z$. Then

$$R_{i2} = T_{i2} = 0, \quad i \neq 2, \tag{A10}$$

$$R_{22} = -\frac{B^2 - b^2}{B^2 + b^2} \to -1, \tag{A11}$$

$$T_{22} = \frac{2b^2}{B^2 + b^2} \to 0, \tag{A12}$$

for $B/b \to \infty$.

By further considering the conservation of momentum in the $x$ and $y$ directions, it is not difficult to derive the result quoted in Eq. (26).


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