

Quantum Phenomena in Low-Dimensional Systems

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A brief summary of the physics of low-dimensional quantum systems is given. The material should be accessible to advanced physics undergraduate students. References to recent review articles and books are provided when possible.

I. INTRODUCTION

A low-dimensional system is one where the motion of microscopic degrees-of-freedom, such as electrons, phonons, or photons, is restricted from exploring the full three dimensions of our world. There has been tremendous interest in low-dimensional quantum systems during the past twenty years, fueled by a constant stream of striking discoveries and also by the potential for, and realization of, new state-of-the-art electronic device architectures.

The paradigm and workhorse of low-dimensional systems is the nanometer-scale semiconductor structure, or semiconductor “nanostructure,” which consists of a compositionally varying semiconductor alloy engineered at the atomic scale [1]. Traditionally one would not include naturally occurring low-dimensional entities such as atoms and molecules in the subject of this article, but some of the most exciting recent developments in the field have involved the use of molecules and even biologically important materials and has blurred the boundaries between the subject and other physical and life sciences. In addition, there are systems of great interest in physics, such as high-temperature superconductors, where the effects of reduced dimensionality are believed to be essential, and these too will be regarded as low dimensional. Many of the subjects covered here are central to the currently fashionable fields of nanoscience and nanotechnology [2,3].

The study of low-dimensional quantum phenomena has led to entirely new fields of research, such as the physics of mesoscopic systems, which will be discussed below. And low-dimensional systems have shed new light on the difficult questions of how disorder (impurities, for example) and electron-electron interaction affect a quantum system. In fact, understanding the combined effects of disorder and interactions in condensed matter systems is currently a problem of enormous interest.

How are electrons, say, restricted from moving in three dimensions? The answer is *confinement*. Take, for example, an electron inside a long wire: The positively charged ions in the wire produce an electric field that prevents the electrons from escaping. Often, in fact, one can regard the electrons as being subjected to a hard-wall potential at the wire’s surface. The electronic eigenstates are given by a plane wave running along the wire multiplied by a localized function in the transverse directions. For a range of low energies the eigenstates have the same

transverse eigenfunction and only the plane wave factor changes. This means that motion in those transverse directions is “frozen out,” leaving only motion along the wire.

This article will provide a very brief introduction to the physics of low-dimensional quantum systems. The material should be accessible to advanced physics undergraduate students. References to recent review articles and books are provided when possible. The fabrication of low-dimensional structures is introduced in Section II. In Section III some general features of quantum phenomena in low dimensions are discussed. The remainder of the article is devoted to particular low-dimensional quantum systems, organized by their “dimension.”

II. MAKING LOW-DIMENSIONAL QUANTUM STRUCTURES

The most common method of fabricating low-dimensional structures is by “growing” compositionally graded semiconductor alloys in high-vacuum molecular-beam epitaxy (MBE) machines. Take, for example, the homogeneous alloy $\text{Al}_c\text{Ga}_{1-c}\text{As}$ consisting of a periodic array of arsenic atoms together with a fraction c of aluminum and $1 - c$ of gallium. The special cases of $c = 0$ and $c = 1$ correspond to the crystalline semiconductors GaAs and AlAs, each with a distinct band structure. The microscopic potential produced by the alloy $\text{Al}_c\text{Ga}_{1-c}\text{As}$, although not strictly periodic, may be regarded as producing a band structure interpolating between that of GaAs and AlAs. In particular, the energy gap between the valence and conduction bands varies with c .

Structures that confine electrons are made by changing the aluminum fraction c during crystal growth, leading to a compositionally graded alloy of the form $\text{Al}_{c(\mathbf{r})}\text{Ga}_{1-c(\mathbf{r})}\text{As}$, where c varies spatially. The resulting band structure variation produces a spatially varying conduction band minimum. Hence, an electron added to the conduction band through doping, optical excitation, or electrical injection, sees a position-dependent potential. By varying c appropriately, one can engineer confining potentials that restrict electron motion to fewer than three dimensions.

In practice, however, it is possible to vary c in one direction only, resulting in, at best, a two-dimensional system. To complete the construction of a semiconductor nanostructure it is often necessary to follow growth with

lithography, the selective etching of prepared surfaces. After coating a surface with a protective material, the “resist,” patterns are imprinted on the resist in a photographic processes with focused light, electron beams, or even atoms. After the imprinted pattern is removed chemically, the underlying semiconductor is etched away, leaving an environment that confines electrons both in the MBE-growth direction and laterally (perpendicular to the growth direction). Another common way to produce lateral confinement is to use lithographic techniques to pattern metallic electrodes, or “gates,” on the surface of a crystal (grown by MBE, for example) that has electrons confined in a buried two-dimensional layer parallel to that surface. By applying voltages to these electrodes, electrons in the layer can be depleted from or attracted to the regions below the electrodes.

Finally, it is in many cases necessary to attach electrical contacts to the electron gases inside these nanostructures and to the metal gates on their surfaces.

The lower limit to the size of the structures one can make is usually determined by the size of the patterns one can make with lithography, which, in turn, is usually determined by the quality of the image formed during the exposure stage. At the time of this writing the semiconductor industry can produce, in an integrated circuit, wires with a thickness of only 10 nm, more than 10 times smaller than the wavelength of visible light.

To make even smaller structures naturally occurring nanometer-scale systems have used. Examples include self-assembled nanostructures and structures incorporating molecules or biological materials. A self-assembly technique to make arrays of few-nanometer quantum dots (small “artificial atoms” where electrons are confined in all three dimensions) by growing a thin layer of highly strained material on top of another crystal has been particularly successful. The strained layer relaxes by breaking into small islands, which form the quantum dots. An excellent example of the use of molecules to make low-dimensional systems is provided by large carbon sheets that can be rolled into stable hollow spheres, known as Bucky balls, or into carbon nanotubes. It is even possible to put other molecules inside a Bucky ball, to arrange Bucky balls into a molecular crystal, and to electrically contact individual nanotubes. Electrical contacts have also been attached to other molecules, turning them into “transistors.” Polymers and even DNA strands have been used to engineer quantum structures at the nanometer scale.

III. PHYSICS IN QUANTUM SYSTEMS OF REDUCED DIMENSIONS

Physics in low-dimensional systems is often different than in three dimensions. We now discuss some of these differences and the theoretical tools used to understand them.

A. Effective Mass Theory

Electrons in semiconductor nanostructures move in the presence of as many as three fields; the periodic or nearly periodic potentials produced by the atoms in the crystal, fields applied externally, and the electron-electron interaction potential. The atomic potentials, which vary at the few Angstrom scale, are usually varying much more rapidly than the others. In this case, assuming the electronic states have energies near the bottom of the conduction band or near the top of the valence band, there is an extremely useful description whereby the original problem of an electron moving in the presence of the atomic as well as other potentials is replaced by the much simpler problem of an electron, now with a different mass, moving in the presence of the slowly varying fields only [4]. For example, the effective mass of an electron in a GaAs conduction band is about 0.067 times the ordinary bare mass. In general, the effective mass depends on the electron’s propagation direction, and can even vary with position.

B. Density of States

An important distinction between systems with different dimensionality is their density of states $N(\epsilon)$, which is the number of states per unit “volume” L^D in an energy range ϵ to $\epsilon + d\epsilon$, divided by $d\epsilon$. L is the linear size of the system, and D can be either 1, 2, or 3. (The zero-dimensional case has to be treated separately). A simple general formula can be derived to determine the energy dependence of $N(\epsilon)$: Given an excitation (electron, phonon, photon) with dispersion relation $\epsilon(k) \propto |k|^\alpha$, the number of states contained within a D -dimensional sphere of radius k in momentum space is proportional to $(L/2\pi)^D k^D$ and the number per unit volume is therefore proportional to k^D or $\epsilon^{D/\alpha}$. The density of states is evidently the derivative of the latter with respect to energy, so $N \propto \epsilon^{\frac{D}{\alpha}-1}$. Phonons and photons ($\alpha = 1$) in three dimensions have $N \propto \epsilon^2$, and in lower dimensions would have $N \propto \epsilon^{D-1}$ as long as the dispersion remains linear. Electrons ($\alpha = 2$) have a density of states proportional to $\epsilon^{\frac{1}{2}}$, ϵ^0 (i.e., energy independent), and $\epsilon^{-\frac{1}{2}}$ in 3, 2, and 1 dimensions, respectively. The expressions for $D = 2$ and $D = 1$ assume only one transverse mode or subband present. When more are present, these additional branches simply add to $N(\epsilon)$.

C. Mesoscopic Physics

Mesoscopic physics is an exciting new field of science made possible by nanostructures [5]. A mesoscopic system is one that is in some sense between a microscopic and macroscopic system. It is typically much larger than

a few atoms or molecules, yet it is small enough that the degrees-of-freedom (usually electrons) have to be regarded as fully quantum-mechanical. More precisely, a mesoscopic system has a size L that is larger than a microscopic lengthscale a (for example the Bohr radius), yet smaller than the phase-coherence length L_φ , which is the characteristic length beyond which a particle loses phase coherence. L_φ generally depends on the dimension and the temperature of the system, as well as on microscopic details. Thus, in simple terms, a mesoscopic system is one that is larger than microscopic and in which quantum mechanics manifests itself fully.

In what follows we shall discuss a few examples and concepts from mesoscopic physics.

1. Aharonov-Bohm Effect

In 1959 Yakir Aharonov and David Bohm predicted that a magnetic field can influence the quantum-mechanical phase of a charged particles, even if the particles are prevented from entering the region containing the magnetic field. This proposal, although a direct consequence of quantum mechanics, violated everyone's intuition and was extremely controversial at the time. The Aharonov-Bohm effect has been observed hundreds of times and shows up everywhere in the study of mesoscopic systems. Excellent accounts of it are given in quantum mechanics textbooks [6] and review articles [7,8].

The basic idea is that a charged particle moving around a closed loop of circumference L accumulates, along with its kinematical phase $2\pi L/\lambda$, where λ is the deBroglie wavelength of the particle, an additional Aharonov-Bohm phase given by $2\pi\Phi/\Phi_0$, where Φ is the magnetic flux enclosed by the ring and $\Phi_0 \equiv hc/e$ is the quantum of magnetic flux. The Aharonov-Bohm phase changes the energies of charged particles and also can shift quantum interference patterns. But the effect can only be observed if the particles are sufficiently phase coherent.

2. Persistent Currents

A simple example of a mesoscopic effect is as follows: Take a small metal or semiconductor ring of circumference L . The mesoscopic ring can be made using the lithographic techniques described above. The ring is an ordinary conductor, not a superconductor. It seems reasonable and is usually the case that in the absence of a magnetic field the ground state of the ring is non-current-carrying. However, if the phase-coherence length L_φ is larger than L then by threading the ring with a magnetic flux Φ the ground state (or finite-temperature equilibrium state) becomes a current-carrying state, and current will flow around the ring without ever dissipating, even in the presence of disorder. The electrons in the ring are

affected by the magnetic field even though they are never in contact with it, a consequence of the Aharonov-Bohm effect.

Nondissipative current-carrying states can occur in a superconducting ring too, but in the superconducting case the current-carrying state is not, strictly speaking, an equilibrium state, but is instead an extremely long-lived metastable nonequilibrium state.

As the ring becomes larger, the magnitude of the persistent current decreases. One reason is that when L exceeds L_φ the electrons no longer are moving coherently around the ring and the Aharonov-Bohm effect is no longer operative. The second reason is that even if L_φ was always larger than L , the magnitude of the maximum persistent current decreases as $1/L$.

3. Phase-Coherent Transport

Electrons in a mesoscopic conductor (a conductor whose dimensions are smaller than L_φ) move as a wave, not as a particle. The behavior is similar, in fact, to an electromagnetic wave propagating inside a wave guide. This wave-like nature leads to many unusual physical properties [9].

Perhaps the most profound is the origin of resistance in mesoscopic conductors. Ordinarily resistance is caused by *inelastic* collisions of the current-carrying electrons with disorder (impurities and other imperfections), other electrons, and with lattice vibrations (phonons). In a mesoscopic conductor, however, an electron typically travels the entire length of the system without undergoing an inelastic collision. Thus it might seem that there would be no resistance at all. But to measure the resistance in a mesoscopic system one has to attach electrical contacts or leads to it, which are macroscopic. What occurs, then, in a mesoscopic conductor, is that the resistance is caused by a combination of *elastic* scattering of electron waves from disorder and inelastic scattering in the macroscopic leads, although the magnitude of the resistance is determined exclusively by the former.

This phenomena is succinctly described by a formula, originally proposed by Rolf Landauer in 1957, for the conductance G (reciprocal of the resistance) of a mesoscopic system. Landauer's formula is simply $G = |t|^2 e^2/h$, where t is the quantum transmission amplitude for an electron to propagate through the system ($|t|^2$ is the transmission probability). The ratio e^2/h of fundamental constants has dimensions of conductance and is about $1/26k\Omega$. The Landauer formula shows that the conductance of a mesoscopic conductor is directly related to the quantum mechanical probability that an electron can propagate through the system without scattering elastically. It makes no reference to the strength of inelastic scattering that actually causes the energy dissipation normally associated with a resistor.

Another important consequence of the phase-coherent

nature of electron transport in a mesoscopic system is a phenomenon called weak localization. To understand weak localization it is first necessary to understand “ordinary” Anderson localization (sometimes called strong localization), named after Phillip Anderson. Anderson localization is the process whereby the eigenstates of an electron gas in a disordered environment become spatially localized around impurities, causing the system to behave as an insulator instead of a conductor. Weak localization is a very different process that also increases the resistance of a disordered conductor (although it is not strong enough to turn a conductor into an insulator).

It comes from a very special quantum interference effect that occurs in systems with time-reversal symmetry: To find the probability P for an electron to propagate from point \mathbf{r} to point \mathbf{r}' , one has to add the quantum amplitudes A_i for the electron to take all possible paths i , and then calculate the modulus squared, $P = |\sum_i A_i|^2$. (This expression is a consequence of Feynman’s path integral formulation of quantum mechanics, but one can also view it as a generalization of the double-slit interference formula to an infinite number of “slits.”) The cross-terms in this expression are responsible for interference.

Usually when open paths with \mathbf{r} different from \mathbf{r}' are considered, the randomness in the A_i wash out any quantum interference effects. However, there is a special class of paths, closed paths with $\mathbf{r} = \mathbf{r}'$, where interference effects can be important. In systems with time-reversal symmetry (which basically means that there can be no applied magnetic field) there will always be pairs of closed paths and their time-reversed counterparts in the above summation that have the same amplitude. The result is that the probability to go from \mathbf{r} to \mathbf{r} , in other words the probability to *go nowhere* is enhanced by quantum interference effects, and this amounts to a measurable increase in resistance.

4. Dephasing by Electron-Electron Interaction

As explained above, weak localization (and more generally, the Aharonov-Bohm effect) occurs when the electron motion is sufficiently phase-coherent. This fact can be exploited to actually *measure* the phase-coherence length L_φ or phase-coherence time τ_φ , the characteristic time beyond which the electron becomes decoherent, in an electron system. L_φ and τ_φ are simply related to each other and only one needs to be measured. The resistance increase due to weak localization depends on the number of closed paths that contribute to the summation in $P = |\sum_i A_i|^2$, and a given closed path contributes as long as its length L is less than L_φ (if $L > L_\varphi$ the electron would not have the phase coherence necessary to exhibit interference). Thus, the magnitude of the weak localization effect—which can be determined by “turning the effect off” by applying a magnetic field—can be used to infer the value of L_φ or τ_φ .

At low temperatures the dominant mechanism for dephasing is electron-electron scattering: A given electron feels a fluctuating electric field produced by the other electrons that scrambles its phase after some time τ_φ . Theoretically it is predicted that the dephasing *rate* vanishes at low temperature as $\tau_\varphi^{-1} \propto T^\beta$, where β is a positive exponent, but recently Richard Webb and others have given experimental evidence for a *saturation* of τ_φ^{-1} in the $T \rightarrow 0$ limit. The physics of low-temperature dephasing is currently a problem of great controversy and interest.

5. Thouless Energy

Mesoscopic physics research has also led to a profound new discovery about quantum systems in general. In the 1970’s David Thouless and collaborators showed that any quantum system possess an important fundamental energy scale, now called the Thouless energy E_T . The Thouless energy is a measure of how sensitive the eigenstates in a quantum system are to a change in boundary conditions. Specifically, E_T is defined as the energy change of a state at the Fermi energy caused by a change from periodic to antiperiodic boundary conditions. E_T would be zero in a system—for example an insulator—with localized eigenstates, because if the wave functions don’t extend to the boundaries their energies will be independent of boundary conditions. Thouless showed that the dimensionless ratio $g \equiv E_T/\Delta\epsilon$ of E_T with the energy level spacing at the Fermi energy, $\Delta\epsilon$, determines whether the system is a conductor ($g > 1$) or an insulator ($g < 1$). In fact, g is equal to the conductance of the system in units of e^2/h .

D. Integrable Systems

An extremely important theoretical aspect of low-dimensional quantum systems is that there is a large family of one-dimensional models that are exactly solvable or integrable, that is, the exact many-body wave functions and energies can be obtained. This is especially fortunate because perturbation theory, which is needed to develop approximate descriptions of three-dimensional interacting systems, tends to break down and become invalid in low dimensions. One of the first quantum many-body problems to be solved exactly was the one-dimensional spin-1/2 Heisenberg model with antiferromagnetic nearest-neighbor interaction, solved by Hans Bethe in 1931. Later other important quantum many-body problems were solved exactly, including (i) the one-dimensional Bose gas with delta-function interaction; (ii) the one-dimensional Hubbard model, which describes interacting electrons hopping on a lattice; (iii) the Kondo model, which describes noninteracting electrons scattering from a localized magnetic impurity, and (iv) one-

dimensional spin-1/2 antiferromagnets with long-range (inverse square-law) exchange interaction.

The formidable mathematical techniques required to solve these problems exactly is more than compensated for by the rich phenomena exhibited by the solutions and by the new concepts they have introduced into modern theoretical physics [10,11]. For example, the explicit solution of a one-dimensional model of interacting Fermions, known as the Tomonaga-Luttinger model, has excited-state solutions describing noninteracting *bosons*, and no “dressed” electrons or quasiparticles as predicted from the general picture of interacting Fermi systems known as Fermi liquid theory. Understanding this solvable model led in the 1980’s to a new general theory of one-dimensional systems known as Luttinger liquid theory and to a powerful new mathematical technique called bosonization.

E. Fermi Liquid Theory and Beyond

One of the reoccurring themes in low dimensional electron systems is whether or not the system is a Fermi liquid. As mentioned above, this term refers to a specific theory of interacting Fermi systems originally proposed by Lev Landau in 1957, for which (along with his pioneering work on superfluidity) he was awarded the 1962 Nobel prize. The modern definition of a Fermi liquid is a system of fermions (usually electrons) where the effect of electron-electron interactions are sufficiently tame that they can be accurately described with perturbation theory. A Fermi liquid is in this sense “close” to a non-interacting electron gas.

In low dimensional systems, especially one-dimensional systems, Fermi liquid theory and many-body perturbation theory can break down, leading to what is called a non-Fermi liquid. The paradigm for non-Fermi liquids is the Luttinger liquid, which will be discussed further below.

IV. TWO-DIMENSIONAL QUANTUM SYSTEMS

Two-dimensional systems are usually the easiest to make in the laboratory, and, as discussed above, are the starting point for most lower-dimensional systems.

A. Surface Science

The study of atomically clean surfaces has emerged as a major area of modern condensed matter physics and chemistry [12]. Topics of current interest include the growth of surfaces (for example, by molecular-beam epitaxy) the physical and chemical properties of atoms adsorbed on surfaces, the physical and electronic structure of the surfaces themselves, and the transitions between

their many phases as external parameters such as temperature, pressure, and coverage are varied. Most bulk phenomena, such as superconductivity and magnetism, have surface counterparts that are interesting in their own right. Surface science is blessed with an abundance of surface-sensitive probes that have permitted extremely detailed experimental studies.

B. Heterostructures

Semiconductor heterostructures, which are simply atomically abrupt interfaces between two different semiconductors, can be thought of as building blocks for more elaborate nanostructures [13]. The most important quantities characterizing them are the conduction band and valence band offsets, which determine the height of the potential steps or discontinuities, seen by electrons and holes. They can be classified into two categories: In type I heterostructures both electrons and holes are attracted to the same side of the interface whereas in type II they are attracted to opposite sides.

A single undoped heterojunction cannot do much; it has no free charge carriers and has a bulk-like DOS. However, by adding donor impurities to the high-energy or barrier side of the heterostructure, electrons will escape to the low-energy side of the interface, leaving the then-ionized impurity centers behind. The dipolar charge distribution formed from the electrons and the ionized donors creates an electric field that confines the electrons to the interface and bends the conduction band (and all bands) into a triangular shape. If the density of electrons is low enough only the lowest transverse mode of the heterostructure will be occupied, resulting in a two-dimensional electron gas (see below). This technique of modulation doping, a term that applies anytime the dopant atoms are placed away from the region where the electrons or holes reside, minimizes the scattering of charge carriers from the ionized impurities, and leads to systems with exceptionally high mobility.

C. Quantum Wells and Superlattices

Putting two heterostructures together makes a quantum well, which, when undoped, looks like the one-dimensional square-well potential familiar from elementary quantum mechanics. The depth of the well in the conduction band is equal to the conduction band offset, and the same holds for the valence band. The main difference is that the motion is unrestricted in the two directions perpendicular to the growth direction. Thus, single-particle eigenstates are labelled by a transverse mode quantum number for the motion in the growth direction, an in-plane wave vector, and a spin projection. The transverse modes form overlapping energy bands called

subbands, because they are bands within the conduction band. As electrons are added to the quantum well its shape changes because of the potential produced by the carriers themselves, but the properties remain qualitatively unchanged. At low enough densities, only the lowest transverse mode (or subband) is occupied, and the motion again becomes two-dimensional, the motion along the growth direction being completely quenched.

Quantum wells don't have to be square in shape; there is nothing that prevents one from varying the semiconductor's composition during growth to produce other confining potentials. For example, parabolic quantum wells have been made that mimic the familiar one-dimensional harmonic oscillator.

Putting many quantum wells together by alternating layers of well and barrier materials leads to a periodic square-well potential known as a superlattice, which is analogous to a common textbook example of a one-dimensional "crystal" called the Kronig-Penny model [14]. The term superlattice refers to the fact that there are two different lattices present, the microscopic one formed by the atoms and the larger one consisting of the periodic array of quantum wells. The periodic square-well potential leads to the formation of energy bands inside each original band, called minibands, for the same reason that the periodic potential in a crystalline solid forms the original bands in the first place. The energy gaps between the minibands are called minigaps.

D. The Two-Dimensional Electron Gas

The two-dimensional electron gas has made a tremendous impact on modern condensed matter physics, material science, and, to a lesser extent, statistical mechanics and quantum field theory. It lies at the boundary between three-dimensional electron systems, which are generally Fermi liquids, and one-dimensional systems, which are not. In fact, where the two-dimensional electron gas stands in this regard is not known at present, although most indications are that it is a Fermi liquid (or possibly what is known as a marginal Fermi liquid). It also lies at another boundary, that between metals and insulators. The three-dimensional electron gas in the presence of disorder (impurities and defects) can be an electrical conductor or insulator, but in the presence of any disorder a one-dimensional system is always an insulator. Previously it was thought that the two-dimensional electron gas was insulating, but recent experiments have shown that at low enough densities it can in fact be conducting. This and a few of the many other exciting areas of two-dimensional electron gas research are briefly described below.

Electrons in a metal or doped semiconductor are usually described as being in a liquid or gas-like phase, reflecting the absence of long-range positional order. However, in 1934 Eugene Wigner predicted in that an electron gas of low enough density, moving in a uniform background of positive charge, should crystallize. The reason for this is that at low densities the electron-electron interaction energy becomes larger than the kinetic energy, and by forming a crystal the interaction energy is minimized. In three dimensions (and zero magnetic field) the Wigner crystal is predicted to form, at zero temperature, at electron number densities less than $n_{3D} = [\frac{4}{3}\pi(r_s a_B^*)^3]^{-1}$, where $r_s \approx 67$. Here $a_B^* \equiv \kappa \hbar^2 / m^* e^2$ is the effective Bohr radius in a solid with dielectric constant κ and effective mass m^* . The density in ordinary three-dimensional metals is much too high to form a Wigner crystal. In two-dimensions (again zero magnetic field) the critical density is $n_{2D} = [\pi(r_s a_B^*)^2]^{-1}$, with $r_s \approx 37$. In semiconductors the effective Bohr radius can be much larger than that in a vacuum, making the required densities easier to achieve.

A strong magnetic field suppresses the kinetic energy of the electrons, enabling a crystalline state at higher density. The physics of Wigner crystals in strong fields is a rich and interesting subject with a lot of current activity [15].

2. *The Metal-Insulator Transition*

As mentioned above, experiments have observed metallic states and transitions between metallic and insulating states in low-density high-mobility two-dimensional electron systems [16]. This was a great surprise, and the first reports, made in 1994, were initially treated with some skepticism. In 1958 Phillip Anderson showed that a three-dimensional electron gas could undergo a "metal-insulator" transition to an insulating state in the presence of sufficiently strong disorder. The insulating state is said to be localized because the single-particle wave functions become localized in space rather than being extended throughout the solid. Then David Thouless and Franz Wegner introduced the idea that the metal-insulator transition could be regarded as a quantum phase transition and they studied the problem using scaling methods developed for the study of critical phenomena. In 1979, Elihu Abrahams and collaborators developed what is now called the scaling theory of localization, which applies to noninteracting electrons and which predicts no metal-insulator transition in two dimensions. Experiments in the early 1980 were consistent with that prediction.

There is no consensus yet for what the correct theory of two-dimensional disordered systems is. However, because electron-electron interaction was not incorporated

in the 1979 scaling theory of localization, it is reasonable to believe that the metal-insulator transition is driven by interactions, and there are good experimental indications of this as well.

3. The Quantum Hall Effect

The quantum Hall effect is an extremely rich and active area of research that can only be touched on here [17]. There are two effects, called the integral and fractional quantum Hall effects. They occur when the two-dimensional electron gas is placed in a strong perpendicular magnetic field. The eigenstates of noninteracting electrons in such a field forms highly degenerate bands known as Landau levels. The number of degenerate states in each Landau level is equal to the number of flux quanta in the system BA/Φ_0 , where B is the field strength and A is the system area. The Landau levels are separated in energy by the cyclotron energy $\hbar\omega_c$, where $\omega_c \equiv eB/m^*c$. If the magnetic field is strong enough all the electrons in the two-dimensional electron gas can be accommodated in the lowest Landau level. It is convenient to measure the electron density n in units of the density that fills exactly one Landau level; this is called the filling factor ν , which can be written as nhc/eB . The filling factor can be controlled experimentally either by changing the electron density or by changing the magnetic field.

When ν is equal to an integer it is observed (at low temperatures) that the transverse or Hall conductance of the electron system is, to an extremely high precision, equal to $\nu \frac{e^2}{h}$, independent of any microscopic details such as the nature and strength of the disorder. (The transverse conductance is the ratio of the current through the sample divided by the voltage drop across the sample in the direction perpendicular to the current flow). This integral quantum Hall effect was discovered experimentally in 1980 by Klaus von Klitzing and coworkers, for which he was awarded the 1985 Nobel prize in physics. At the same time that the transverse conductance is quantized, the ordinary longitudinal resistance vanishes. This integral quantum Hall effect can be understood to be a consequence of the Landau-level structure of the noninteracting spectrum.

More surprising was the experimental discovery starting in 1982 of states with quantized Hall conductance at certain fractional values of ν , such as $\frac{1}{5}, \frac{2}{9}, \frac{3}{13}, \frac{3}{11}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{7}{9}, \frac{4}{5}$, and so on. In a brilliant 1983 paper Robert Laughlin explained the simplest fractional states ($\nu = 1/3$ and $1/5$) by essentially guessing the exact many-body wave function for the ground and lowest excited states. His work showed that at these special densities the two-dimensional electron system forms a highly correlated liquid-like ground state with an energy gap of order $0.01 \kappa e^2/\ell$, where $\ell \equiv \hbar c/eB$ is the magnetic length. This energy gap has the same effect as the energy gap between Landau levels in the

noninteracting case and leads to a quantized transverse and vanishing longitudinal conductance. Laughlin also predicted that at filling factors close to $\nu = 1/q$, with q an odd integer, the ground states would consist of his correlated state plus quasiparticles or quasiholes having a *fractional* charge of magnitude e/q . Not only are these particles fractionally charged, they obey “fractional” or anyon statistics, intermediate between Bose-Einstein and Fermi-Dirac statistics. Soon after Laughlin’s work, Duncan Haldane and Bertrand Halperin were able to explain most of the other observed fractions by assuming that the quasiparticles themselves form correlated Laughlin states when their densities reach certain values.

Laughlin’s fractionally charged quasiparticles are believed to have been observed experimentally, first by Vladimir Goldman and Bo Su in a 1995 tunneling experiment, and more recently by two groups measuring the shot noise (current fluctuations) in the fractional quantum Hall effect regime. The proposed fractional statistics has not been directly observed yet. Robert Laughlin was awarded the 1998 Nobel prize in physics for his profound work in this area.

There will not be room to do justice to the incredible amount of important work that has been done on this subject. But it is worth pointing out the physical reason why the properties of the two-dimensional electron gas in a strong magnetic field are so unusual and often exotic. The reason is that electron-electron interactions are, in a certain sense, much stronger than in an ordinary metal. A dimensionless measure of the strength of interactions is the ratio of the typical interaction energy, say e^2/r_0 , where r_0 is the average interparticle distance, to the typical kinetic energy of electrons at the Fermi energy. In an ordinary metal the latter would simply be the Fermi energy, but in quantum Hall effect systems where the electrons are in the lowest Landau level ($\nu \leq 1$) there is no real kinetic energy, because all electrons in the lowest Landau level have the *same* energy, which, without loss of generality, can be taken to be zero. In this case the dimensionless ratio is infinite! Thus, the quantum Hall effect system is one where many-body effects are extremely important.

4. Composite Fermions

In 1989 Jainendra Jain proposed a theory unifying the integral and fractional quantum Hall effects [18,19]. The idea is to make an exact transformation whereby an even number of fictitious magnetic flux tubes, each of strength hc/e , are attached to each electron. The resulting composite object (electron plus flux tubes) obeys Fermi statistics and is called a composite Fermion. The Fermi statistics in this case is a consequence of the underlying Fermi statistics of the bare electron combined with the Aharonov-Bohm phase shift produced by the

flux tubes; had an odd number of flux tubes been mathematically attached the resulting object would instead be a composite boson.

The flux-attachment transformation is an exact transformation of the two-dimensional electron gas problem, and the many-body Hamiltonian is now much more complicated than it was originally. However, when a mean field approximation is applied to this transformed Hamiltonian an integral quantum Hall effect of the composite fermions results. This technique provides a method to reliably calculate properties of fractional quantum Hall effect systems. The physics underlying the fractional quantum Hall effect is sufficiently complicated that it cannot be described by a mean field theory for the original Hamiltonian. The magic of the composite Fermion transformation is that it makes an exact transformation of the many-body Hamiltonian to a form where simple analysis becomes adequate.

Even more interesting things happen when the transformation is applied to non-quantized Hall states, for example when $\nu = \frac{1}{2}$. In this case there are two real flux tubes per electron in the system. Now two fake flux tubes are attached, oriented opposite to the real ones, to each particle. On average, then, the fictitious magnetic field cancels the real field, leaving no magnetic field left. Composite fermion theory then predicts that the two-dimensional electron gas, although in a very strong perpendicular magnetic field, should behave as though there were *no* field at all. This astonishing effect has been observed in a series of beautiful experiments.

E. High-Temperature Superconductors

The phenomena of superconductivity, and the related phenomena of superfluidity, are extremely interesting subjects that are covered in a number of excellent introductory texts [20,21]. Superconductivity was discovered in 1911 by Kamerlingh Onnes, who observed an abrupt transition to a resistance-free state in mercury near 4 K, and, since then, there has been a search for materials that become superconducting at higher temperatures. In 1986 a new class of oxides were discovered by George Bendorz and Karl Müller that have superconducting transition temperatures as high as 130 K. Bendorz and Müller were awarded the 1987 Nobel prize in physics for this discovery.

These high-temperature superconductors have weakly connected two-dimensional sheets composed of Cu and O, and most theories of these materials assert that the physics responsible for the superconductivity is a consequence of these CuO₄ planes. Thus, in this sense, high-temperature superconductors are also examples of low dimensional quantum systems.

There is no consensus yet as to the microscopic origin of high-temperature superconductivity, and a discussion of the competing proposals is beyond the intended level

of this article. However, a few introductory statements can be made.

First, the superconducting state itself seems to be of the same type as in conventional superconductors, where electrons (or holes) acquire an attractive interaction causing them to bind into bosonic pairs which then Bose condense. This description of superconductivity, now called BCS theory, was developed by John Bardeen, Leon Cooper, and Robert Schrieffer, for which they won the 1972 Nobel prize in physics. Beautiful experiments measuring the spontaneous magnetization of YBa₂Cu₃O₇ rings have determined that the pair wave function has *d*-wave symmetry, the same symmetry as a *d* orbital.

When undoped, the CuO layers are antiferromagnetic insulators. Electrons removed from them allow in-plane conduction by holes. As a hole moves around the surrounding spins must constantly readjust themselves to lower their antiferromagnetic exchange energy. It turns out that the exchange energy can be kept lower if the holes move together. Thus, the background antiferromagnetism is at least partly responsible to the pairing of the holes.

The non-superconducting state of these novel materials has turned out to be at least as unusual as the superconducting state. For example, it is not even clear whether the normal states are Fermi liquids or non-Fermi liquids. Certain common features in these materials, for example the temperature dependence of the resistivity, have no explanation to date.

F. Two-Dimensional Magnetism

The discovery of high-temperature superconductors with antiferromagnetic CuO planes has also stimulated new interest in two-dimensional magnetism, especially in two-dimensional Heisenberg antiferromagnets.

The Mermin-Wagner theorem, which states that a continuous symmetry (in this case, rotational symmetry) cannot be broken in a two-dimensional system at finite temperature, requires that magnetism in the two-dimensional Heisenberg model exist, if at all, at zero temperature only.

Whether or not there is long-range order at $T = 0$ depends on the parameters in the Hamiltonian, and this leads to an interesting situation: By varying parameters in the Hamiltonian, the quantum ground state can change, say, from paramagnetic to antiferromagnetic. This type of transition is called a *quantum* phase transition because it is analogous to thermodynamic phase transitions between phases that occur as the temperature is varied, but here the temperature is always zero.

In contrast with the Heisenberg model, a finite-temperature magnetic state of the two-dimensional Ising ferromagnet, which does not have continuous symmetry, does exist.

The study of low-dimensional quantum magnetism continues to be an exciting and active area of research. Sophisticated mathematical and computational techniques have been developed to extract their physical properties [22].

V. ONE-DIMENSIONAL QUANTUM SYSTEMS

One-dimensional systems possess some of the most exotic phases of condensed matter. In these systems, the properties of even weakly interacting particles differ dramatically from that of noninteracting ones, and the generic conducting state of one-dimensional bosons, fermions, or even spins is a Luttinger liquid instead of a Landau Fermi-liquid.

A. Quantum Wires

Quantum wires are extremely narrow wires where electron motion is allowed in one direction, along the wire, but confined in the other two directions. Most often they are created by putting metallic gates on top of a two-dimensional electron gas and applying voltages to deplete the electron gas underneath. This can produce quantum wires of varying length whose width can be controlled during an experiment. In fact, varying the width changes the number of transverse modes that have energies below the Fermi energy and that contribute to the conductance of the wire. The quantum wires come complete with electrical contacts, made from the underlying two-dimensional electron gas, which allow transport measurements to be made.

The Landauer formula, described above, predicts that the conductance of such a wire is equal to $2e^2/h$ times the number of transverse modes below the Fermi energy. Thus, as the width of the wire is varied, the conductance is expected to be quantized in units of $2e^2/h$. By “quantized conductance” one means that the conductance is equal to $2e^2/h$ times an integer, and that integer increases as the wire is made wider.

This has been observed in many pioneering experiments, but recent experiments (on wires made using the cleaved-edge-overgrowth technique) show quantization in multiples of approximately 0.7 times that which is expected. These intriguing experiments are not understood at this time.

B. Carbon Nanotubes

Nature has provided beautiful molecular quantum wires called carbon nanotubes, formed by rolling carbon sheets of hexagonal symmetry into tubes of varying diameter [23]. They can have diameters as little as a nanometer, and can be produced as isolated single-wall

tubes, isolated multi-wall tubes, or in bundles or ropes of parallel tubes.

Both their structural and electronic properties are unusual. They are predicted to be the strongest fibers known, and do not break when bent. In fact, the tube wall appears to simply straighten out after the bending forces are released. Great progress has been made in obtaining a microscopic understanding of the structural properties of carbon nanotubes.

Much is also known about the transport properties of nanotubes; however, there remains many unanswered questions. The sheets of carbon making up the tubes, when undoped, are insulators. When they are rolled into tubes they can be metals or insulators, depending on the tube diameter and the amount of twist, if any, introduced during their creation. Actually, it is surprising that a nanotube, if they are regarded as a one-dimensional crystal, can be conducting at all, because one-dimensional metallic crystals are usually unstable to the formation of a periodic lattice distortion or strain that lowers their energy but makes them insulating. Their tubular geometry, however, is believed to make them sufficiently resistant to longitudinal strain.

Then there is the question of electron-electron interaction effects, which are usually very important in one-dimensional systems. Luttinger liquids (see below) have been predicted to occur in nanotubes, complicated by the fact that the conducting channels (for each spin orientation) come in pairs. Electron-phonon interaction is also believed to be strong in nanotubes, and there is even the possibility of turning them into superconductors.

C. The Luttinger Liquid State

In the 1950’s and 1960’s Sin-itiro Tomonaga and Jaoquin Luttinger studied theoretical models of one-dimensional metals (similar to quantum wires) and found that they possessed properties entirely different than three-dimensional metals and different than that expected by Landau’s Fermi liquid theory. In particular, the low-energy eigenstates described elementary excitations that are bosons, not fermions. These bosons are basically phonons (or quanta of the sound waves) of the electron gas.

The breakdown of Fermi liquid theory in one dimension is now known to be the norm, and the generic non-Fermi-liquid state of matter in one dimension is called the Luttinger liquid. The simplest type of Luttinger liquid is characterized by a single dimensionless parameter g , which, roughly speaking, characterizes the degree to which the system deviates from a Fermi liquid, defined as $g = 1$. The value of g is determined by the parameters characterizing the electron gas, like its density and electron-electron interaction strength.

Luttinger liquids are interesting because of their exotic behavior, including an unusual conductivity as a func-

tion of temperature, voltage, and frequency, and their extreme sensitivity to disorder. Experimental efforts to observe them in the laboratory have been unsuccessful until only very recently.

D. Edge States of the Quantum Hall Fluid

The most obvious place to look for a Luttinger liquid is in a quantum wire. This has been attempted for years without success because it has not been possible to make clean enough quantum wires and disorder ruins the Luttinger liquid state, making it insulating. The quantum Hall state of a two-dimensional electron gas, however, has provided an alternative.

The electrical current present in an ordinary metal (when connected to a battery, for example) is just the sum of the current carried by each electron. At the same time, when the metal is in its ground or equilibrium state there is no net current flowing (assuming a macroscopic conductor so there is no persistent current). This means that in thermodynamic equilibrium the currents from all the electrons present must sum to zero. Accordingly, to describe the current-carrying state of a metal it is sufficient to keep track of *changes* of the state occupation numbers from their equilibrium values.

In the linear-response regime, where the metal is only slightly out of equilibrium, only the occupation numbers near the Fermi energy change. This is why transport properties like resistance are referred to as “Fermi surface” properties.

Now, in the quantum Hall effect system, when the conditions are such that a quantized Hall conductance is observed, the states at the Fermi energy are physically located at the edges of the two-dimensional electron gas. (In the bulk of the system, away from the edges, the Fermi energy is in an energy gap—either between Landau levels or in the energy gap of many-body origin described by Laughlin.) As in an ordinary metal, to describe the current-carrying state it is sufficient to keep track of the states at the Fermi energy, which are called “edge states.” The edge state concept has been extremely helpful in understanding strong-field magnetotransport in the quantum Hall effect regime and in other low-dimensional electron systems [24].

Edge states corresponding to *fractional* quantum Hall effect states are especially interesting, because they are predicted to be Luttinger liquids. However, there some important distinctions between these Luttinger liquids (called chiral Luttinger liquids) and the Luttinger liquids described above: First, the value of g in a chiral Luttinger liquid is predicted to be simply related to the filling factor ν of the electron gas, so it can be determined rather easily. Second, g can be made to differ appreciably from 1, making the associated non-Fermi-liquid effects larger. And third, edge states can be made to be entirely insensitive to disorder [24].

Luttinger liquid behavior has been observed in quantum Hall effect systems, but the values of g measured do not generally agree with the theoretical predictions. This is another interesting unsolved problem in the physics of low-dimensional quantum systems.

E. Spin Chains

Spin chains are one-dimensional lattices of spins. Although they occur in nature, embedded in three-dimensional materials such as $\text{Mn}(\text{HCOO})_2 \cdot 2\text{H}_2\text{O}$, CsNiCl_3 , and RbNiCl_3 , they are also of great theoretical interest because they exhibit dramatic disorder and quantum fluctuation effects that occur in higher dimensions as well. One-dimensional spin systems also have a large class of exactly solvable versions and can be efficiently studied numerically, giving them certain advantages over higher-dimensional magnets.

One of the most interesting quantum-fluctuation effects in spin chains was discovered by Duncan Haldane in 1983. In higher-dimensional antiferromagnets it is believed that the ground states can be antiferromagnetically ordered and support a gapless spin-wave spectrum. Although true long-range order is prohibited in one dimension, quasi-long-range order, where the spin-spin correlation functions decay at large distances as a power-law, is allowed, and it was assumed that antiferromagnetic spin chains also had gapless spin-wave excitations. What Haldane predicted, and what was later established experimentally and by numerical studies, was that this picture was correct for spin chains with half-integer spins ($\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$) but that integer spin system ($1, 2, 3, \dots$) have finite energy gaps.

VI. ZERO-DIMENSIONAL QUANTUM SYSTEMS

Zero-dimensional systems are confined in all three directions, and, as such, have discrete energy spectra (isolated energy levels separated by gaps).

A. Quantum Dots

Quantum dots are nanometer-scale structures, often made from gated and compositionally graded semiconductors, that can hold a few to a few thousand electrons [25,26]. In many respects they are like artificial atoms, the inverse-square-law Coulomb force of the nucleus being replaced by a linear, harmonic-oscillator-like force provided by the confinement. However, unlike natural atoms, the number of electrons in a quantum dot can be precisely controlled and varied during an experiment, allowing, in some sense, one to “sweep” through the periodic table and beyond.

Like other nanostructures, quantum dots are studied with both transport measurements and optical spectroscopy, with no magnetic field and in large fields. They are interesting both from a fundamental physics point-of-view and for their potential applications. Applications include transistors that control currents at the single-electron level, highly efficient low-power lasers, and even circuit elements for quantum computers (see below).

1. Coulomb Blockade and the Single Electron Transistor

A simple yet fascinating effect occurs when one tries to pass current through a quantum dot: From a classical point-of-view, transport through the dot has to occur via a discrete change of charge of the dot, because the total charge of N electrons in the dot is $-Ne$, where $-e$ is the charge of one electron. But this means that the electrostatic energy $(Ne)^2/2C$, where C is the capacitance of the quantum dot (approximately equal to its diameter), changes by a finite amount, which, for small enough dots, can be larger than the thermal energy $k_B T$ and the energy supplied by the battery eV . Here k_B is Boltzmann's constant, T is the temperature, and V is the applied voltage. In this situation no current can flow, a phenomena called Coulomb blockade because it originates from Coulomb interaction between electrons.

This simple classical picture is not entirely correct, because quantum mechanically an electron can tunnel from one lead to the other, through the quantum dot, violating energy conservation on the dot for a short period of time. However, this is a small effect (although interesting in itself) for a quantum dot very weakly connected to leads.

A novel transistor can be made by adding a third lead to the quantum dot. Unlike the first two leads, however, the third lead is not connected to the dot itself, but rather, to a metallic gate near the dot. Varying the voltage V_G on this gate changes the electrostatic energy of the electrons in the dots, and can be used to lower the Coulomb blockade barrier.

The transistor works as follows: In the presence of the metallic gate the classical electrostatic energy (when there are N electrons in the quantum dot) is $E = N^2 e^2 / 2C - eNV_G$. For a given value of V_G the optimum number of electrons on the dot (the number that would minimize E) is $V_G C / e$. However, $V_G C / e$ is generally not an integer, so N takes the value of the integer closest to $V_G C / e$. As the gate voltage is continuously varied, then, the number of electrons in the dot and the number passing through the leads changes one at a time. We therefore have a transistor operating at the level of single electrons!

A number of other interesting phenomena have been observed in quantum dot systems. A beautiful example is the recent observation of the Kondo effect, which is a low-temperature many-body effect where the spins of

the conduction electrons interact very strongly with the spins of the electrons in the dot, enhancing the conductance of the dot when there is an odd number of electrons in it. Because quantum dots are somewhat like atoms, several classic atomic physics phenomena have also been observed in them.

2. Quantum Computers

Quantum dots in the near future may even become circuit elements for a revolutionary type of computer.

Quantum computing is a new multidisciplinary subject of great current interest to university researchers, and also of great importance to government agencies and the information and technology industries [27]. The term "quantum computing" refers to the possibility of building a computer out of elements, called quantum logic gates, whose operation exploits the laws of quantum mechanics to perform operations prohibited by conventional or classical logic gates. There is also a related subject called quantum information science in which the quantum properties of matter are to quantify, store, encode, and communicate information. Recent theoretical and experimental breakthroughs in these subjects have attracted computer scientists, mathematicians, physicists, and chemists [28].

Whereas the basic unit of information in a classical computer is a binary digit or bit, a "0" or a "1" say, a quantum computer processes information in the form of a coherent superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, called a quantum bit or qubit. Thus, the elementary component in a quantum computer is the two-level system familiar from quantum mechanics.

There are two major efforts in quantum computing research. One emphasis is to assume that one has a number of qubits that can be manipulated, and ask what they can be used for. Are there calculations that a quantum computer can do much faster than a classical computer? The answer is yes: If a quantum computer could be built it could factor large numbers that would be essentially impossible to factor with a classical computer using the best known algorithms. And this assumed impossibility of factoring large integers is the basis for current encryption methods used by the military and by banks.

The other emphasis is try to design and build a quantum computer. The qubit referred to above is of course an abstract two-level system. What is the best physical realization of the qubit? Certainly one wants a qubit with a long phase-coherence time τ_ϕ (see above), because this is the time during which the system remains quantum mechanical and during which a quantum computation can be performed. Second, one has to figure out how to manipulate qubits and have them interact to perform an actual calculation. Electron spin states in quantum dots have emerged as excellent candidate qubits.

Quantum computing is very new and it is hard to pre-

dict if it will be successful in the near term. But if a quantum computer is successfully built, it will be revolutionary.

3. Exact Diagonalization

A theoretical technique that is particularly well suited to study electrons in quantum dots is the direct numerical solution of the many-body Schrödinger equation. This is usually done by expanding in a basis of noninteracting eigenstates, so the computational problem becomes that of finding the eigenvalues and eigenvectors of huge matrices, often called exact diagonalization. It is not truly exact because it involves a truncation of the original infinite-dimensional Hilbert space, but this can often be done in a controlled manner, yielding results that are insensitive to that truncation.

Exact numerical studies of quantum many-body systems are usually impossible because it is only possible to handle a dozen or so particles. But small quantum dots can be studied experimentally, making exact diagonalization methods extremely useful.

B. Artificial Molecules

Another new area of research is the study of *pairs* of quantum dots, which then can be thought of as artificial molecules. Just as in natural molecules, bonds can be formed between the atoms by electrons shared between dots. These systems permit controlled studies of (at least some) basic molecular phenomena. One problem of current interest is the magnetic properties of artificial molecules.

C. Nanocrystals and Nanoparticles

There is considerable interest in nanometer-scale crystals containing, say, from 10^3 to 10^6 atoms [29]. These can have structural, electronic, and optical properties very different than bulk solids. Spherical or nearly spherical nanocrystals are sometimes called nanoparticles.

Semiconducting and insulating nanocrystals have many similarities to quantum dots. In fact, the electronic properties are often the same, because in both cases the electron is confined in all three directions. However the vibrational properties can be quite different: A quantum dot is not usually mechanically isolated from a bulk solid, so it supports a continuum of low-frequency vibrational modes. But an isolated nanocrystal would have a *discrete* vibrational spectrum, much like that of a drum head. This means that any physical property involving phonons will be very different than that in a macroscopic solid.

Metallic nanocrystals have also been actively investigated. They can be made out of superconductors, allowing for the study of mesoscopic superconductivity. Making nanocrystals out of magnetic materials leads to fascinating new magnetic properties that may have useful data-storage applications. Metallic nanoparticles also have unusual optical properties because they are easily polarizable, and the propagation of an electromagnetic wave through a cluster of metallic nanoparticles probes the electromagnetic modes of that cluster.

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