Universal quantum simulation with pre-threshold superconducting qubits: Single-excitation subspace method

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We propose a method for general-purpose quantum computation and simulation that is well suited for today’s pre-threshold-fidelity superconducting qubits. This approach makes use of the n-dimensional single-excitation subspace (SES) of a system of n tunably coupled qubits. It can be viewed as a nonscalable special case of the standard gate-based quantum computing model, but allows many operations in the unitary group SU(n) to be implemented by a single application of the Hamiltonian. Our approach bypasses the need to decompose the evolution operator into elementary gates, making large, nontrivial computations possible without error correction. The method is especially well suited for universal quantum simulation, specifically simulation of the Schrödinger equation with a real but otherwise arbitrary \(n \times n\) Hamiltonian. We argue that a 1000-qubit SES processor, which would require no known improvements in superconducting device technology and which could be built today, should be capable of achieving quantum speedup relative to a petaflop supercomputer. We speculate on the utility and practicality of such a universal quantum simulator.

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I. INTRODUCTION AND MOTIVATION

A. The quest for quantum speedup

In the standard gate-based model of quantum computation with \(n\) qubits, the initial state of the (closed) system is represented by a 2\(^n\)-dimensional complex vector \(\psi\), and the computation is described by unitary time-evolution operator \(U\). Running the quantum computer implements the map

\[\psi \rightarrow U\psi.\]  

A key feature here is the exponential classical information storage capacity of the wave function \(\psi\). However, the number of one- and two-qubit gates required to implement an arbitrary element of the unitary group SU(2\(^n\)) is at least \(2^{2n} - 1\), making the construction of an arbitrary \(U\) inefficient (using elementary gates). The primary goal of quantum algorithm design is to implement interesting cases of \(U\) with a polynomial number of elementary gates (it is also necessary that the desired answer be obtained with high probability upon measurement, possibly after a polynomial number of repetitions). The class of problems that can be solved with success probability > \(\frac{1}{2}\) by a quantum circuit with depth bounded by a polynomial in the length of the input is called BQP. Two important algorithms in the BQP complexity class are the quantum algorithms for order finding \cite{Shor1994} and weak simulation of local (or sparse) Hamiltonians \cite{Nielsen2010}. Although these algorithms are technically efficient, the number of gates required for interesting applications is still large \cite{Stamper2006} and an error-corrected quantum computer is therefore required.

A major milestone for the field of quantum computation will be to achieve quantum speedup relative to the most powerful classical supercomputer (itself a rapidly moving target). Here we assume a general-purpose or universal quantum computer, in contrast with analog quantum simulators and other special purpose approaches where the simulated Hamiltonian or problem type is restricted to a certain family. Analog quantum simulation \cite{Ladd2010} is already a highly developed and successful approach to quantum computation, and only 50 or so degrees-of-freedom become classically intractable \cite{Knill2008} (fewer for analog open-system simulators \cite{Alicki2001}, where decoherence is considered part of the problem to be simulated). In fact, the break-even point, where the classical and quantum computation times are equal, has probably already been achieved with analog simulators. We will not attempt to identify these examples here, which would require a sharp degree-of-controllability criterion to distinguish a “proper” analog quantum simulation from a more traditional experiment, which also has a number of control parameters and which can rarely be simulated classically. Currently, the best performing approach to universal fault-tolerant quantum computation (in terms of threshold value and hardware scalability) appears to be the surface code \cite{Fowler2012}. Conservative estimates of the number \(n\) of physical qubits required for break-even with a 99.9% fidelity (superconducting) surface code quantum computer range from \(10^5\) for quantum simulation to
10^7 for factoring. Even running a long, error-corrected computation with a only a few logical qubits appears to require more than a thousand physical qubits. Thus, the break-even point for fault-tolerant universal quantum computation is likely to take some time to achieve.

B. Pre-threshold quantum computation

The single-excitation subspace (SES) method described here is not intended to replace the standard model of scalable, fault-tolerant quantum computation outlined above. Rather, it provides an alternative approach to general-purpose quantum computation that can be implemented with today’s pre-threshold hardware, especially superconducting circuits. We use the term “pre-threshold” to refer to qubits and architectures with insufficient fidelity to enable fault-tolerant quantum computation. State-of-the-art demonstrations of quantum algorithms with pre-threshold qubits are necessarily restricted to no more than about 10 qubits [25–36]. With superconducting circuits, this restriction is not a consequence of a limitation of fabrication (it is already possible to make devices with hundreds of qubits), but instead that the available coherence time inhibits their application to larger problem sizes, which would require larger circuit depth and longer computation times. The SES method bypasses the need to use one- and two-qubit gates, allowing larger computations to be carried out within the available coherence time.

Table I compares three broad approaches to quantum computation and simulation. The left column lists attributes achievable by a universal, fault-tolerant quantum computer. The single-excitation subspace (SES) method described here is not intended to replace the standard model of scalable, fault-tolerant quantum computation outlined above. Rather, it provides an alternative approach to general-purpose quantum computation that can be implemented with today’s pre-threshold hardware, especially superconducting circuits. We use the term “pre-threshold” to refer to qubits and architectures with insufficient fidelity to enable fault-tolerant quantum computation. State-of-the-art demonstrations of quantum algorithms with pre-threshold qubits are necessarily restricted to no more than about 10 qubits [25–36]. With superconducting circuits, this restriction is not a consequence of a limitation of fabrication (it is already possible to make devices with hundreds of qubits), but instead that the available coherence time inhibits their application to larger problem sizes, which would require larger circuit depth and longer computation times. The SES method bypasses the need to use one- and two-qubit gates, allowing larger computations to be carried out within the available coherence time.

### TABLE I. Three approaches to pre-threshold quantum computation and simulation

<table>
<thead>
<tr>
<th>Attribute</th>
<th>Small system</th>
<th>SES method</th>
<th>Analog/spec purp</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalable</td>
<td>×</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>Universal</td>
<td>✓</td>
<td>✓</td>
<td>×</td>
</tr>
<tr>
<td>Speedup</td>
<td>×</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>Arb accuracy</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>Arb runtime</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
</tbody>
</table>

The SES method can be applied to a wide variety qubit-qubit interaction types (see appendix), but without loss of generality we restrict ourselves to the simple σx ⊗ σx coupling of [2]. Although we assume an architecture based on superconducting qubits, our approach might apply to other architectures as well.

The FQC model [2] is usually considered to be unscalable, because of the O(n^2) tunable coupling circuits and wires, a position that we also adopt here [37]. In gate-based universal quantum computation, the FQC and LQC models are actually equivalent in the sense that any quantum circuit implemented by a FQC can be implemented by a LQC after adding chains of SWAP gates, which only introduce polynomial overhead. However, this equivalence is restricted to the standard gate-based approach and does not apply here.

### II. QUANTUM COMPUTATION IN THE SES

#### A. Hardware model

Consider the following model of an array of n coupled superconducting qubits,

\[
H_{FQC} = \sum_i \epsilon_i c_i^\dagger c_i + \frac{1}{2} \sum_{ii'} g_{ii'} \sigma_i^x \otimes \sigma_{i'}^x,
\]

written in the basis of uncoupled-qubit eigenstates. Here \(i, i' = 1, 2, \ldots, n\), and

\[
c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

The \(\epsilon_i\) are qubit transition energies and \(g_{ii'}\) are qubit-qubit interaction strengths; both are assumed to be tunable. \(g_{ii'}\) is a real, symmetric matrix with vanishing diagonal elements. We also require microwave pulse control of at least one qubit, and simultaneous readout (projective measurement of \(\sigma^\dagger\)) of every qubit. Model [2] describes a fully connected network of qubits, which we refer to as a fully connected quantum computer (FQC). The FQC model should be contrasted with local quantum computer (LQC) models that have coupling only between nearby qubits (nearest neighbors, for example). The SES method can be applied to a wide variety qubit-qubit interaction types (see appendix), but without loss of generality we restrict ourselves to the simple \(\sigma^x \otimes \sigma^x\) coupling of [2]. Although we assume an architecture based on superconducting qubits, our approach might apply to other architectures as well.

#### B. Device parameters

Superconducting qubits have been reviewed in Refs. [33–41]. Although the model [2] can be realized with several qubit designs, the transmon qubit [42] currently has the best performance. (We note that frequency tunability is required here, which will increase sensitivity to flux noise). For concreteness we assume a qubit frequency \(\epsilon/2\pi\) in the range 5 - 6 GHz and coupling strength \(g/2\pi\) in the range −100 to 100 MHz.
FIG. 1. Superconducting phase qubit and tunable inductive coupler circuits.

FIG. 2. (color online) SES processor layout.

C. Processor layout

A fully connected quantum computer of \( n \) qubits operating in the SES mode requires \( n(n - 1)/2 \) coupler circuits and associated wires. A variety of tunable couplers can be used for this purpose \[43-48\]. For definiteness, we consider the tunable inductive coupler \[48\] demonstrated with superconducting phase qubits and show a processor layout that avoids excessive crossovers.

Circuit diagrams for a single phase qubit “\( q \)” and single coupler “\( c \)” are illustrated in Fig. 1, where the crossed boxes represent Josephson junctions. In terms of these elements, a possible layout for a fully connected \( n = 5 \) array is shown in Fig. 2. If we assume a qubit size of 10\( \mu \)m \( \times \) 10\( \mu \)m and a 10nm spacing of the coupler wires (blue vertical lines in Fig. 2), then a 1000-qubit processor would require about a 1cm \( \times \) 1cm chip.

D. Single-excitation subspace

The idea we explore here is to perform a quantum computation in the \( n \)-dimensional single-excitation subspace of the full \( 2^n \)-dimensional Hilbert space. This is the subspace spanned by the computational basis states

\[ |\psi\rangle = \sum_{i=1}^{n} a_i |i\rangle, \quad \sum_{i=1}^{n} |a_i|^2 = 1, \quad (5) \]

which corresponds to a point on the sphere \( S^{2n-1} \). For example, the states \( |\text{unif}\rangle \) include the maximally entangled \( W \)-type state

\[ |\text{unif}\rangle = \frac{|1\rangle + |2\rangle + \cdots + |n\rangle}{\sqrt{n}} = \frac{|10\cdots0\rangle + |01\cdots0\rangle + \cdots + |00\cdots1\rangle}{\sqrt{n}}. \quad (6) \]

E. SES Hamiltonian

The advantage of working in the SES can be understood from the following expression for the SES matrix elements of model \( \mathcal{H}_FQC \),

\[ \mathcal{H}_{ii'} \equiv \langle i|\mathcal{H}_{FQC}|i'\rangle = \epsilon_i \delta_{ii'} + g_{ii'}. \quad (7) \]

Because the diagonal and off-diagonal elements are directly and independently controlled by the qubit energies and coupling strengths, respectively, we have a high degree of control over the SES component of the quantum computer’s Hamiltonian. This property allows many \( n \)-dimensional unitary operations to be carried out in a
single step, bypassing the need to decompose into elementary gates, and also enables the direct quantum simulation of real but otherwise arbitrary time-dependent Hamiltonians (a preliminary account of this application is given in Ref. [42]).

There are two reasons why the SES Hamiltonian (7) is not completely controllable:

1. \( \mathcal{H}_{ii'} \) is real, whereas the most general Hamiltonian is complex Hermitian. The experimentally available control parameters, consisting of \( n \) qubit energies and \( n(n-1)/2 \) coupling strengths, are sufficient to control \( n(n+1)/2 \) independent parameters of an \( n \times n \) symmetric matrix that is real.

2. There are experimental limitations on the range of values that the \( \epsilon_i \) and \( g_{ii'} \) can take. We define \( \epsilon \text{max} \) to be the magnitude of the largest coupling available in a particular experimental realization (a realistic value is 100 MHz).

We will leave the discussion of possible generalizations to complex Hamiltonians for future work.

III. APPLICATIONS

A. Entangled state preparation

It is simple, for example, to generate the entangled state \( |\psi\rangle \) in a single step: Consider the \( n \times n \) real Hamiltonian

\[
\mathcal{H} = g \begin{pmatrix}
2 & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

(8)

where qubit 1 is symmetrically coupled to all other qubits, which are themselves uncoupled (a star network). Only two of the eigenfunctions—call them \( |\psi\pm\rangle \)—have overlap with \( |1\rangle \), so the evolution of the \( |1\rangle \) state is effectively a two-channel process. The spectrum is as follows: States \( |\psi\pm\rangle \) have energy \( \epsilon \pm = g(1 \pm \sqrt{n}) \); all other eigenfunctions are degenerate with energy \( \epsilon = 0 \). Evolution for half a period corresponding to the splitting \( 2\sqrt{n}g \), namely

\[
t_{\text{qu}} = \frac{\pi}{2\sqrt{n}g},
\]

leads to the desired operation

\[
|\text{unif}\rangle = \exp\left[-i\frac{\pi}{2\sqrt{n}} \begin{pmatrix}
2 & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\right]|1\rangle.
\]

(10)

This can be implemented in a few ns with superconducting circuits.

B. Grover’s search algorithm

Although we have argued that the SES method is probably most useful for universal quantum simulation, it is also interesting to consider a traditional quantum algorithm. Here we apply the SES method to Grover’s search algorithm for a single marked state \( |i\rangle \) in a database of size \( n \), which takes the form

\[
(WO_i)^K |\text{unif}\rangle \approx |i\rangle, \quad \text{with} \quad K = \frac{\pi}{4\sqrt{n}}.
\]

Here

\[
W \equiv 2|\text{unif}\rangle \langle \text{unif}| - I
\]

(11)

\[
O_i \equiv \begin{pmatrix}
1 & & & & \\
& 1 & & & \\
& & \ddots & & \\
& & & -1 & \\
& & & & 1
\end{pmatrix}
\]

(13)

is the inversion operator,

\[
\mathcal{H} = g \begin{pmatrix}
2 - n & 2 & \cdots & 2 \\
2 & 2 - n & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots \\
2 & 2 & \cdots & 2 - n
\end{pmatrix},
\]

(12)

which describes a completely symmetric, fully coupled array, for a time

\[
t_{\text{qu}} = \frac{\pi}{ng}.
\]

(15)

This leads to the desired operation

\[
W = \exp\left[-i\frac{\pi}{n}\begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
1 & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 0
\end{pmatrix}
\right],
\]

(16)

up to a multiplicative phase factor. The oracle \( O_i \) can be generated by a \( 2\pi \) rotation on qubit \( i \). Each iteration of the search can be implemented in a few ns, allowing SES processors of even modest sizes to perform computations that would otherwise require thousands of elementary gates.
C. Time-independent Schrödinger equation solver

Finally, we consider wave function propagation by a real but otherwise arbitrary time-independent \( n \times n \) Hamiltonian \( H \),

\[
|\psi\rangle \rightarrow e^{-iHt}|\psi\rangle.
\]  
(17)

Here \( t \) is the length of simulated time (for example, the duration of some physical process to be simulated). To map this to the SES processor we first find the smallest positive constant \( \lambda \) such that every matrix element of

\[
H_{\text{qc}} \equiv \frac{H - \text{const} \times I}{\lambda}
\]
(18)
is between \(-g_{\text{max}}\) and \( g_{\text{max}}\). With the SES processor we then simulate the equivalent evolution

\[
|\psi\rangle_{\text{qc}} \rightarrow e^{-iH_{\text{qc}}t_{\text{qc}}} |\psi\rangle_{\text{qc}},
\]
(19)

where

\[
t_{\text{qc}} \equiv \lambda t.
\]
(20)

The time \( t_{\text{qc}} \) required to perform a single run of the quantum computation is therefore

\[
t_{\text{qc}} = t_{\text{qc}} + t_{\text{meas}},
\]
(21)

where \( t_{\text{meas}} \) is the (projective) qubit measurement time, which for superconducting qubits we can assume to be about 100 ns \[51\].

How long does the corresponding classical simulation take? This depends on the particular Hamiltonian, through the value of \( \lambda \). However, it is possible to assess the possibility of quantum speedup by finding the time required to classically simulate a typical run of the SES processor, where \( H_{\text{qc}} \) is an \( n \times n \) random symmetric matrix with elements bounded by \( g_{\text{max}} \). The optimal algorithm used to perform this classical simulation depends on the value of \( t_{\text{qc}} \): When \( t_{\text{qc}} \) is less than a parameter \( t^* \) that depends on \( n \), the fastest way to classically simulate \[19\] is to solve the Schrödinger equation

\[
\partial_{t}|\psi\rangle_{\text{qc}} = -iH_{\text{qc}}|\psi\rangle_{\text{qc}}
\]
(22)
as a set of \( n \) coupled ordinary differential equations. In this regime the classical simulation time scales linearly with \( t_{\text{qc}} \). When \( t_{\text{qc}} > t^* \) the fastest way to simulate \[19\] is by matrix exponentiation or diagonalization, and the classical simulation time is independent of \( t_{\text{qc}} \).

Next we specialize to the case

\[
n = 1000 \quad \text{and} \quad \frac{g_{\text{max}}}{2\pi} = 100 \, \text{MHz}.
\]
(23)

In this case it can be shown that (with a single core)

\[
t^* \approx 1 \, \text{ns},
\]
(24)

and in what follows we consider an evolution with \( t_{\text{qc}} \) longer than \( t^* \). The result \[24\] is only weakly dependent on the particular algorithms used to integrate the ordinary differential equations and diagonalize the Hamiltonian. Although a precise value of \( t^* \) is not needed here, it is important that it is sufficiently less than the coherence time of the quantum computer. The classical (single core) simulation time is then equal to the matrix exponentiation or diagonalization time, which is about 1s for a \( 1000 \times 1000 \) real, symmetric matrix. Thus we would obtain speedup relative to a single core if we can perform a 1000 qubit simulation, and read out the result, in less than about 1s.

Our objective is to achieve speedup relative to a state-of-the-art supercomputer, not a single core. The classical simulation time (matrix diagonalization time) then needs to be evaluated on a supercomputer, using an optimally distributed parallel algorithm. However, we can bound the parallel performance by using the single-core result and assuming perfect parallelization efficiency. If we approximate a petaflop supercomputer by \( 10^6 \) gigaflop cores, we conclude that the classical calculation time can be no shorter than \( 10^{-6} \) times the single-core time. For the \( n = 1000 \) case considered above the classical simulation is therefore no shorter than \( t_{\text{cl}} = 1 \, \mu \text{s} \), while the quantum simulation (including readout) can be performed in a few hundred ns. Thus we would achieve a genuine quantum speedup for Schrödinger evolution with a 1000-qubit SES simulator.

IV. CONCLUSIONS

It is tempting to conclude that by working in an exponentially small (approximately invariant) subspace of the full \( n \)-qubit Hilbert space, any computational power that quantum physics has over classical is sacrificed, but this is not the case. Any BQP computation necessarily accesses only exponentially small subspaces (polynomially larger than the SES, but still exponentially small) \[52\]. This is because a quantum computer starts in a product state and then implements a polynomial number of two-qubit entangling gates.

It is also interesting to compare the space resource requirement (number of physical qubits) of the SES method relative to a fault-tolerant quantum computer. Because approximately \( 10^4 \) physical qubits are required per ideal qubit in the surface code (this includes 10 logical qubits per ideal for magic state distillation), the logical Hilbert space simulated has dimension

\[
\sim 2^{10^{-4}n},
\]
(25)

which is smaller than \( n \) when \( n < 1.7 \times 10^5 \). This shows that for a superconducting quantum computer with fewer than about a hundred thousand physical qubits, an SES processor will simulate a higher dimensional Hilbert space than the surface code will, although the fault-tolerant computer will of course run for a much longer time.
We have argued here that the 1000-qubit SES quantum simulator should be capable of speedup relative to a petaflop supercomputer. However, there are important limitations of the SES method:

1. We require a fully connected network with tunable coupling between every qubit, which therefore has \( n(n - 1)/2 \) coupling circuits and wires. This may make the SES approach unscalable beyond a few thousand qubits.

2. We require an exponentially large number of physical qubits relative to the standard model. However, because of the large physical qubit overhead required by error correction, the SES method will be simulating a larger Hilbert space until there are about a hundred thousand physical qubits.

3. The maximum computation time is limited by the coherence time of the hardware and calculations cannot run indefinitely. This applies to all known pre-threshold approaches, including analog quantum simulators.

4. Computations have fixed accuracy and cannot be made arbitrarily accurate. This also applies to all known pre-threshold approaches, including analog simulators. Error sources include decoherence, leakage out of the SES, readout errors, as well as pulse control errors.

5. In the present SES approach the simulated Hamiltonian must be real.

6. In our estimates of classical simulation time, we have not included the time needed to store the Hamiltonian in memory or perhaps compute it from a separate procedure. Similarly, for the quantum simulation time estimates we have not included the time required to send the \( n(n + 1)/2 \) voltage signals to the qubits and couplers before the simulation, and to reset the readout circuitry afterwards.

7. It is important to emphasize that the SES method implements a weak simulation (providing a single sample from the distribution of possible measurement outcomes), not the probability distributions themselves as is normally provided classically. This is of course also a limitation of the conventional approaches designed for error-corrected quantum computers, and is usually a limitation of analog quantum simulation as well.

8. We have argued here that a quantum speedup is possible with 1000 qubits, but not every 1000-qubit simulation will possess this speedup (this depends on the particular simulated Hamiltonian, the value of \( \lambda \), and the desired accuracy).

9. The SES method is unsuitable for applications (including Shor’s) requiring enormous Hilbert spaces.

We note, however, that in contrast to efficient gate-based simulation intended for a scalable quantum computer, the SES approach is not restricted to local (or sparse) Hamiltonians.

The quantum simulation application discussed here is restricted to time-independent Hamiltonians. The time-dependent case, discussed previously in Ref. [49], is perhaps even more interesting. However, the estimates of both \( t_{ee} \) and \( t_{q} \) in the time-dependent case are somewhat more subtle, and for the purpose of establishing speedup it is simpler to treat the time-independent case.

The SES method appears to be distinct from previously investigated approaches to quantum computation. Like analog quantum simulation, it is tied to a very specific hardware model and cannot be implemented on any architecture. However, it enables (almost) universal quantum computation and simulation, and might make practical a first-generation universal quantum simulator capable of achieving genuine quantum speedup.

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**Appendix A: General coupling types**

Here we discuss the generalization of the SES method to FQC models of the form

\[
H = \sum_i \epsilon_i c_i^\dagger c_i + \frac{1}{2} \sum_{ii'} g_{ii'} \sum_{\mu\nu} J_{\mu\nu} \sigma_i^\mu \otimes \sigma_i'^\nu, \quad (A1)
\]

where the \( \sigma^\mu \) (with \( \mu = x, y, z \)) are Pauli matrices and \( J_{\mu\nu} \) is a fixed, real, dimensionless tensor determined by the hardware. In this case the SES matrix elements are

\[
(i|H|i') = \left[ \epsilon_i - 2\left( \sum_j g_{ij} J_{zz} + \left( \sum_{j<j'} g_{jj'} \right) J_{zz} \right) \delta_{ii'} + \left( J_{xx} + J_{yy} - i(J_{xy} - J_{yx}) \right) g_{ii'} \right]. \quad (A2)
\]

Note that the term proportional to \( \sum_{j<j'} g_{jj'} \delta_{ii'} \) is an energy shift and can be dropped. The results given in the main body of the paper apply when

\[
J_{xx} + J_{yy} \neq 0 \quad (A3)
\]

and

\[
J_{xy} = J_{yx}. \quad (A4)
\]

The condition (A3) means that the interaction has an exchange or transverse component, and (A4) ensures that \( g_{ii'} \) is purely real.
Further generalizations are possible, including cases where the coupling introduces both single-qubit and two-qubit terms, and cases where $g_{ij}$ is complex, but these will not be considered here.

[36] This is a matter of definition, of course, and we note that in practice a reasonable definition of hardware scalability is perhaps that the manufacturing cost is a linear or polynomial function of qubit number $n$. (However, actual manufacturing costs of even LQC models are likely to be more complex functions consisting of plateaus separated by large discontinuities.) Nevertheless, as scalability is not essential to our approach we prefer to avoid any distractions resulting from this question and simply declare the FQC model to be unscalable.