The University of Georgia
Department of Physics and Astronomy

Prelim Exam

January 7, 2022

Part I (problems 1, 2, 3, and 4)
9:00 am – 1:00 pm

Instructions:

- Start each problem on a new sheet of paper. Write the problem number on the top left of each page and your pre-arranged prelim ID number (but ***not*** your name) on the top right of each page.
- Leave margins for stapling and photocopying.
- Write only on *one side* of the paper. Please ***do not*** write on the back side.
- If not advised otherwise, derive the mathematical solution for a problem from basic principles or general laws (Newton's laws, the Maxwell equations, the Schrödinger equation, etc.).
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- Show your work and/or explain your reasoning in *all* problems, as the graders are not able to read minds. Even if your final answer is correct, not showing your work and reasoning will result in a *substantial* penalty.
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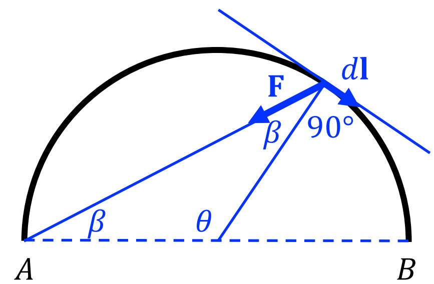
Problem 1: Classical Mechanics

A particle moves along a semicircular path of radius R from one end (point A) to the other end (point B). At every point along its path, the particle feels a force of constant magnitude F_0, but always directed toward its starting location, point A. How much work does the particle have to do against this force to go from A to B?



The work done by the field, W_{field}, and the work done by an external agent against the field, W_{ext}, have to be equal and opposite if there is no change in the kinetic energy of the particle: $W_{\text{ext}} = -W_{\text{field}}$. The work done by the field is:

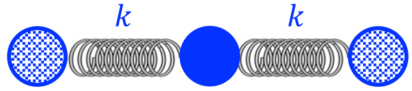
$$W_{\text{field}} = \int_{\text{path}} \mathbf{F} \cdot d\mathbf{l}.$$

From the figure, $\mathbf{F} \cdot d\mathbf{l} = F_0 \cos(\beta + 90°)\, dl$. If we parametrize the path by θ, then θ goes from 0° to 180°, $dl = R d\theta$, and $2\beta + \theta = 180°$ so that $\beta + 90° = 180° - \theta/2$. Hence, $\mathbf{F} \cdot d\mathbf{l} = F_0 \cos(180° - \theta/2)\, R d\theta = -F_0 R \cos(\theta/2)\, d\theta$, and:

$$W_{\text{ext}} = -W_{\text{field}} = \int_{0°}^{180°} F_0 R \cos(\theta/2)\, d\theta = 2F_0 R \sin(\theta/2)\big|_{0°}^{180°} = 2F_0 R.$$

Problem 2: Classical Mechanics

Three masses $m_1 = m$, $m_2 = 2m$, and $m_3 = m$ are constrained to slide freely in one dimension on a horizontal frictionless track. Masses m_1 and m_2 are connected by a spring of force constant is k, and masses m_2 and m_3 are also connected by a spring of force constant k. (a) How many normal modes are there for this system? (b) What are the frequencies of the normal modes?



(a) There are three masses that can each move in one dimension. Hence, there are three degrees of freedom and thus three normal modes. Two of these are vibrational modes and one is a zero-frequency, pure translational mode.

(b) The equations of motion are:

$$\begin{aligned} m_1\ddot{x}_1 &= k(x_2 - x_1) \\ m_2\ddot{x}_2 &= k(x_1 - x_2) + k(x_3 - x_2) \\ m_3\ddot{x}_3 &= k(x_2 - x_3). \end{aligned}$$

Normal modes are those for which all atoms oscillate at the same frequency; i.e., $x_i = x_{i0}e^{i\omega t}$, for $i = 1, 2, 3$. Upon substitution, the equations above become:

$$\begin{aligned} m_1\omega^2 x_{10} &= m\omega^2 x_{10} = k(x_{10} - x_{20}) \\ m_2\omega^2 x_{20} &= 2m\omega^2 x_{20} = k(-x_{10} + 2x_{20} - x_{30}) \\ m_3\omega^2 x_{30} &= m\omega^2 x_{30} = k(-x_{20} + x_{30}). \end{aligned}$$

Dividing the middle equation by 2, we can cast this as an eigenvalue equation:

$$\begin{bmatrix} k & -k & 0 \\ -k/2 & k & -k/2 \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix} = m\omega^2 \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix} \equiv \lambda \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix},$$

with eigenvalues $m\omega^2$. The eigenvalues of the k-matrix are found as follows:

$$\begin{vmatrix} k-\lambda & -k & 0 \\ -k/2 & k-\lambda & -k/2 \\ 0 & -k & k-\lambda \end{vmatrix} = 0$$

$$(k-\lambda)^3 + 0 + 0 - 0 - \tfrac{1}{2}k^2(k-\lambda) - \tfrac{1}{2}k^2(k-\lambda) = 0$$

$$(k-\lambda)^3 - k^2(k-\lambda) = 0$$

$$\lambda = k \quad \text{or} \quad k - \lambda = \pm k \quad \Longrightarrow \quad \lambda = 0, k, \text{or } 2k.$$

Hence, the normal-mode frequencies are:

$$\omega = 0 \quad \text{or} \quad \omega = \sqrt{\frac{k}{m}} \quad \text{or} \quad \omega = \sqrt{\frac{2k}{m}}.$$

Problem 3: Electromagnetism

A spherical conductor of radius a carries a charge Q. It is surrounded by a linear dielectric material of susceptibility χ_e out to radius b. What is the energy of this configuration?

The electrostatic energy, W, of a configuration of charge is given by:

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau,$$

where $\mathbf{D}$ is the electric displacement field and $\mathbf{E}$ is the electric field. For the configuration given in the problem, a simple Gauss's Law calculation gives:

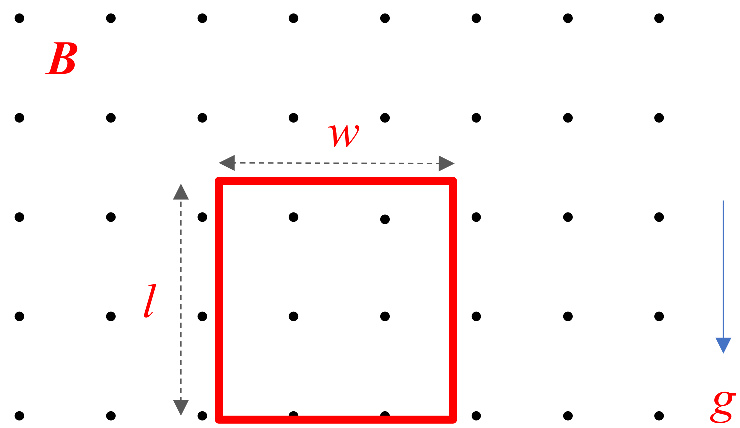
$$\mathbf{D}(r) = \begin{cases} 0, & r < a \\ \dfrac{Q}{4\pi r^2}\hat{\mathbf{r}}, & r > a \end{cases} \quad \text{and} \quad \mathbf{E}(r) = \begin{cases} 0, & r < a \\ \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & a < r < b \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & r > b \end{cases}$$

Hence:

$$\begin{aligned} W &= \frac{1}{2}\frac{Q^2}{(4\pi)^2}\cdot 4\pi\left[\frac{1}{\epsilon}\int_a^b \frac{1}{r^2}\frac{1}{r^2}r^2\,dr + \frac{1}{\epsilon_0}\int_b^\infty \frac{1}{r^2}\frac{1}{r^2}r^2\,dr\right] \\ &= \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{\epsilon}\left(-\frac{1}{r}\right)_a^b + \frac{1}{\epsilon_0}\left(-\frac{1}{r}\right)_b^\infty\right] \\ &= \frac{Q^2}{8\pi}\left[\frac{1}{1+\chi_e}\left(\frac{1}{a}-\frac{1}{b}\right) + \frac{1}{b}\right] \\ &= \frac{Q^2}{8\pi\epsilon_0}\frac{1}{1+\chi_e}\left[\frac{1}{a}+\frac{\chi_e}{b}\right]. \end{aligned}$$

Problem 4: Electromagnetism

A thin conducting rectangular frame with mass M and resistance R is released from the rest and falls from a region of constant magnetic field B normal to the frame into a region of zero field. Find the velocity of the frame.



According to Faraday's Law, a frame moving with a speed v out of a region of uniform magnetic field can generate an EMF:

$$\varepsilon = vBw.$$

Therefore, there is a counter-clockwise induced current in the frame:

$$i = \frac{\varepsilon}{R} = \frac{vBw}{R}.$$

This induced current feels a magnetic force for the horizontal edge of the frame still inside the magnetic field. This force opposes gravity:

$$\mathbf{F}_{\text{mag}} = iwB\hat{\mathbf{z}} = -\frac{B^2w^2v}{R}\hat{\mathbf{z}},$$

where

$$\mathbf{F}_{\text{grav}} = -Mg\hat{\mathbf{z}}.$$

According to Newton's 2nd Law, the equation of motion of the frame is:

$$M\frac{dv}{dt} = -\frac{B^2w^2v}{R} - Mg.$$

If we let $v_\infty = gMR/B^2w^2$, the equation above can be written as:

$$\frac{dv}{dt} + \frac{g}{v_\infty}v = -g.$$

The solution to this differential equation is:

$$v(t) = v_\infty\left(e^{-gt/v_\infty} - 1\right).$$

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Part II (problems 5 and 6)
2:00 pm – 4:00 pm

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Problem 5: Quantum Mechanics

Consider a free particle in one dimension.

(a) Show that the stationary states of the Hamiltonian are of the form:

$$\Psi(x,t) = Ae^{i[kx-(\hbar k^2/2m)t]}.$$

Identify k in terms of the energy E of the free particle. Do not bother with any normalization!

Applying the Hamiltonian to $\Psi(x,t)$ yields:

$$\hat{\mathcal{H}}\Psi = \frac{\hat{p}^2}{2m}\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = \frac{\hbar^2 k^2}{2m}\Psi = E\Psi\,.$$

Since $\Psi(x,t)$ satisfies the time-independent Schrödinger equation, it is an energy eigenstate (and therefore a stationary state) with energy:

$$E = \frac{\hbar^2 k^2}{2m}.$$

The factor $e^{-i(\hbar k^2/2m)t}$ clearly has the form $e^{-iEt/\hbar}$, which means that $\Psi(x,t) = \Psi(x,0)e^{-iEt/\hbar}$ is a time-dependent stationary state.

[*Note*: This has the form of $E = p^2/2m$, where $p = \hbar k$.]

(b) Determine the probability current j_x of this wave function. In what direction does the current flow?

Hint: If you do not remember the formula for the probability current, you can derive it from the continuity equation (the time change of the probability density must equal the negative gradient of the probability current), combining it with the time-dependent Schrödinger equation. In one dimension:

$$\frac{\partial}{\partial t}|\psi(x,t)|^2 + \frac{\partial j_x}{\partial x} = 0.$$

The probability current is:

$$j_x = \frac{\hbar}{2mi}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{dx}\right)$$
$$= \frac{\hbar}{2mi}(ik+ik)\Psi^*\Psi = \frac{\hbar k}{m}|A|^2.$$

Since $j_x > 0$, the current flows in the $+x$ direction.

Problem 6: Quantum Mechanics

A spin-1 particle is in a state of orbital angular momentum $l = 2$.

(a) What are the possible values of j, the total angular momentum quantum number?

The total-angular-momentum quantum number j can take on values from $|l - s|$ to $l + s$ in integer steps, where l and s are the orbital-angular-momentum and spin-angular-momentum quantum numbers, respectively. Since $l = 2$ and $s = 1$:

$$j = 1, 2, \text{or } 3.$$

(b) List the states of the uncoupled basis, $\{|lsm_l m_s\rangle\}$. How many $|lsm_l m_s\rangle$ are there?

Given l and s, the allowed values of m_l and $m_s$are $-l, -l+1, \dots, l-1,$ and l and $-s, -s+1, \dots, s-1, s$, respectively. Hence, there are $2l + 1$ values of m_l and $2s + 1$ values of m_s, so that the total number of states is $(2l + 1)(2s + 1)$. For the case $l = 2$ and $s = 1$, the $5 \cdot 3 = 15$ states of the uncoupled basis are:

$$\begin{aligned}\{|lsm_l m_s\} = \{&|2,1,2,1\rangle;\ |2,1,2,0\rangle;\ |2,1,2,-1\rangle;\ |2,1,1,1\rangle;\ |2,1,1,0\rangle;\\ &|2,1,1,-1\rangle;\ |2,1,0,1\rangle;\ |2,1,0,0\rangle;\ |2,1,0,-1\rangle;\ |2,1,-1,1\rangle;\\ &|2,1,-1,0\rangle;\ |2,1,-1,-1\rangle;\ |2,1,-2,1\rangle;\ |2,1,-2,0\rangle;\ |2,1,-2,-1\rangle\}.\end{aligned}$$

(c) List the states of the coupled basis, $\{|jm_j\rangle\}$. How many $|jm_j\rangle$ are there?

There are the same number of states in the coupled basis. For $l = 2$ and $s = 1$, the 15 states of the coupled basis are:

$$\begin{aligned}\{|jm_j\rangle\} = \{&|3,3\rangle;\ |3,2\rangle;\ |3,1\rangle;\ |3,0\rangle;\ |3,-1\rangle;\ |3,-2\rangle;\ |3,-3\rangle;\\ &|2,2\rangle;\ |2,1\rangle;\ |2,0\rangle;\ |2,-1\rangle;\ |2,-2\rangle;\ |1,1\rangle;\ |1,0\rangle;|1,-1\rangle\}\end{aligned}$$

(d) Express the coupled basis state $|jm_j\rangle = |32\rangle$ in the uncoupled basis, $\{|lsm_l m_s\rangle\}$, by lowering the state $|jm_j\rangle = |33\rangle$.

Since $m_j = m_l + m_s$, there is one and only one uncoupled basis state that can contribute to the $|jm_j\rangle = |3,3\rangle$ coupled basis state; namely:

$$|3,3\rangle = |2,1,2,1\rangle.$$

We now apply $\hat{J}_- \equiv \hat{L}_- + \hat{S}_-$ to both sides. Lowering the left side gives:

$$\hat{J}_-|3,3\rangle = \hbar\sqrt{3(3+1) - 3(3-1)}|3,2\rangle = \sqrt{6}\hbar|3,2\rangle.$$

Lowering the right side gives:

$$(\hat{L}_- + \hat{S}_-)|2,1,2,1\rangle = \hat{L}_-|2,1,2,1\rangle + \hat{S}_-|2,1,2,1\rangle = 2\hbar|2,1,1,1\rangle + \sqrt{2}\hbar|2,1,2,0\rangle.$$

Equating the two yields:

$$\sqrt{6}\hbar|3,2\rangle = 2\hbar|2,1,1,1\rangle + \sqrt{2}\hbar|2,1,2,0\rangle$$

$$\therefore \quad |3,2\rangle = \sqrt{\frac{2}{3}}\,|2,1,1,1\rangle + \frac{1}{\sqrt{3}}\,|2,1,2,0\rangle.$$