Department of Physics and Astronomy University of Georgia

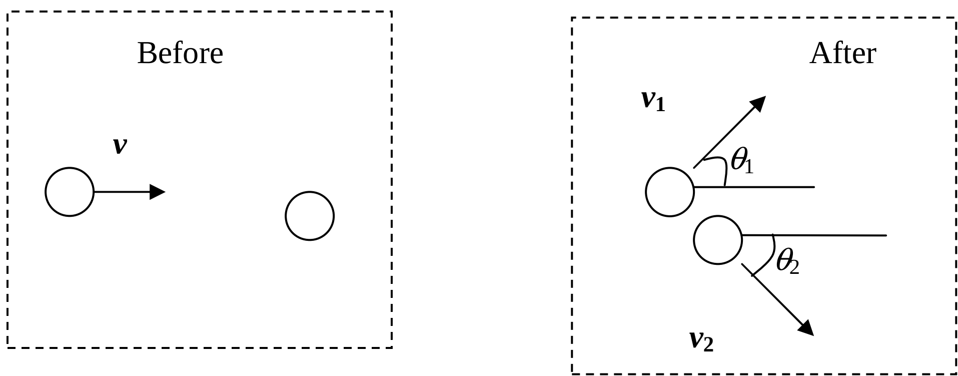
January 2008 Written Comprehensive Exam

— Day 1 —

This is a closed-book, closed-note exam. You may use a calculator, but only for arithmetic functions (not for doing algebra or for referring to notes stored in memory). Attempt all six problems. Start ***each problem*** on ***a new sheet of paper*** (not merely on a new side) and use one side only. Print your name on each piece of paper that you submit. For full credit you must show your work and/or explain your answers.

PROBLEM 1 (one part)

Consider a collision between two objects of equal mass, *m*, on a frictionless surface, one of which is initially at rest. Let the collision be a glancing collision as shown in the figure. Show that if the two objects, moving such that after the collision their velocity vectors make a 90^o angle with respect to one another, that the collision is elastic.



PROBLEM 2 (three parts)

An object of mass *m* is moving in a circle of radius *R*. It's position as a function of time ***t*** is described by the angular coordinate, $\theta = At + Bt^2$, where *A* and *B* are constants

(a) Find expressions, in terms of *m*, *R*, *A* and *B*, for the time dependence of
  the tangential speed, v_T
  the tangential acceleration a_T
  and the radial acceleration a_R

(b) Find the time from t = 0 for the object to complete one complete revolution

(c) Obtain the work done on the mass in completing this revolution.

PROBLEM 3 (one part)

The electric potential of some charge distribution is given by, $\Phi(\vec{r}) = A\frac{e^{-\lambda r}}{r}$, where A and λ are constants; find the charge density $\rho(r)$.

PROBLEM 4 (two parts)

(a) Let the Hermitian operator $\hat{Q}$ represent a quantum observable. Show that the expectation value of $\hat{Q}$ in *any* quantum state satisfies the following equation:

$$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle,$$

where $\hat{H} = \hat{T} + \hat{V}$ is the Hamiltonian of the system.

(b) Suppose the system is initially prepared in an energy eigenstate: $\Psi(x,\ 0) = \phi(x)$, where $\hat{H}|\phi\rangle = E|\phi\rangle$. Analyze the operator $\hat{Q} = \hat{x}\hat{p}$ to derive the *Virial Theorem*:

$$2\langle T\rangle = \left\langle x\frac{\partial V}{\partial x}\right\rangle.$$

PROBLEM 5 (two parts)

A particle of mass m is in a normalized quantum mechanical state, $\psi(x,t) = Ae^{-\beta[(mx^2/\hbar)+it]}$, where A and β are positive real constants.

(a) Find A.

(b) For what potential energy function $V(x)$ does ψ satisfy the Schrödinger equation?

PROBLEM 6 (two parts)

The internal energy of a particular substance is given by $U = Ap^2Vn$, where p is the pressure, V is the volume, n is the number of moles, and A is a positive constant. Suppose one mole of this substance in mechanical equilibrium (*i.e.*, zero pressure) has a volume of V_0.

(a) How much pressure is required to *adiabatically* compress one mole of the substance from V_0 to $\frac{1}{4}V_0$? [Hint: You must first find a relationship between p and V for an adiabatic process.]

(b) How much work was done on the system in part (a)?

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— Day 2 —

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PROBLEM 1 (one part)

Consider a simple harmonic oscillator. Calculate the time averages of the kinetic and potential energies over one cycle.

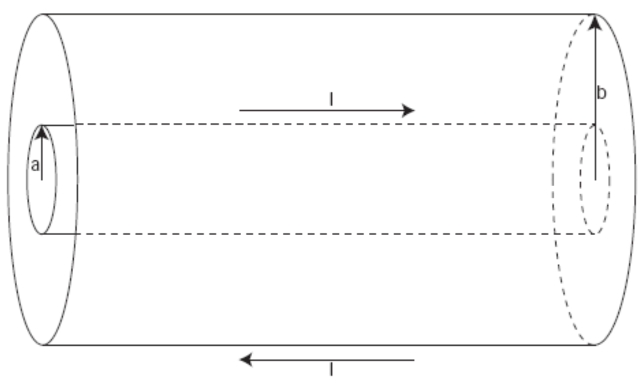
PROBLEM 2 (three parts)

A long coaxial cable carries a current I, *i.e.* the current flows down the surface of the inner cylinder, radius a, and back along the outer cylinder, radius b, as shown in the figure below. The two conductors are separated by vacuum. Neglecting the fringe effect,

(a) Find the magnetic field $\vec{B}$.

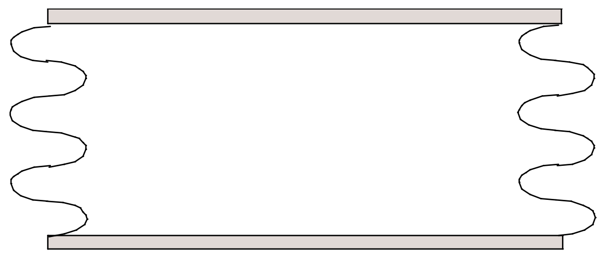
(b) Find the magnetic energy stored in a length l.

(c) Find the self inductance L.



PROBLEM 3 (one part)

Two stiff, 50cm long parallel wires are connected at their ends by conductive springs (see figure) and placed horizontally on a frictionless table. Each spring has an un-stretched length of 5.0 cm and a spring constant of 0.025 N/m. The wires push each other apart when a current travels around the loop. How much current is required to stretch the springs to lengths of 6.0 cm?



PROBLEM 4 (five parts)

Two quantum observables, $\hat{A}$ and $\hat{B}$, on a three-dimensional Hilbert space are expressed as matrices in the basis of $\hat{A}$ eigenkets by:

$$\hat{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \text{ and } \hat{B} = \begin{pmatrix} 0 & b & 0 \\ b & 0 & b \\ 0 & b & 0 \end{pmatrix},$$

where a and b are positive real numbers. Suppose the system is prepared in some quantum state, $|\psi\rangle$, represented as a column matrix in the basis of $\hat{A}$ eigenkets by:

$$|\psi\rangle = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

(a) Are $\hat{A}$ and $\hat{B}$ compatible observables? Justify your answer.

(b) Determine the allowed measurable values for the observable $\hat{B}$.

(c) If a measurement of $\hat{A}$ were made for the system in state $|\psi\rangle$, what would be the probability of obtaining $A = a$? What would be the (normalized) state of the system immediately following this measurement?

(d) Suppose the measurement in (c) actually *does* give $A = a$. If a measurement of $\hat{B}$ were made immediately afterward, what would be the probability of obtaining $B = 0$? What would be the (normalized) state of the system immediately following this measurement?

(e) Finally, suppose the measurement in (d) actually *does* give $B = 0$. If $\hat{A}$ were measured again immediately afterward, what would be the probability of obtaining $A = a$? What would be the (normalized) state of the system immediately following this measurement?

PROBLEM 5 (one part)

A point source of light is placed above a thick plate of glass with index of refraction n. The distance from the source to the upper surface of the glass is L and the thickness of the glass is D. A ray of light from the source may suffer either single reflection at the upper surface, or single reflection at the lower surface, or multiple alternating reflections at the lower and upper surfaces. Thus, each ray splits into several rays, giving rise to multiple images. In terms of L, D and n, find the distances of the first and second images below the surface of the plate. The first image is that formed by direct reflection from the upper surface; the second image is formed by a single reflection at the lower surface of the glass. Assume that the angle of incidence of the ray is small and that the index of refraction of air is 1.

PROBLEM 6 (two parts)

For the following problem, you may find it useful to utilize the following Lorentz transformation equations relating the positions and times of events measured by observes in two reference frames, S and S', moving at a speed v along the positive x-axis relative to S.

$$x' = \gamma(x - vt)\,; \quad t' = \gamma(t - vx/c^2)\,; \quad y' = y\,; \quad z' = z\,; \quad \gamma = (1 - v^2/c^2)^{-1/2}$$

(a) An observer at rest relative to a barn is observing a stick traveling through his barn at a speed, v, that is a large fraction of the speed of light (*i.e.* it is relativistic). He notes that the front end of the stick leaves the barn at the same time as the back end enters the barn. The barn and stick have proper lengths of L_B and L_S, respectively. Find an expression, in terms of v and L_B for the time difference that an observer at rest relative to the stick measures for the two events, namely the front end of the stick exiting the barn and the back end entering it. Which event occurs first according to observers in the reference frame of the stick?

(b) A muon is observed to be traveling at a relativistic speed, u', relative to the observer in the rest frame of the stick. The muon is moving in a direction parallel to the relative velocity of the two reference frames. Derive an expression for the speed, u, of the muon relative to the reference frame of the barn in terms of u' and v.

Note Sheet for Spring 2008 Qualifying Exam

<u>Vector Identities</u>: (**a**, **b**, **c** and **d** are vector fields and ψ is a scalar field)

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$	$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$
$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{d})$	
$\nabla\times\nabla\psi=0$	$\nabla\cdot(\nabla\times\mathbf{a})=0$
$\nabla\times(\nabla\times\mathbf{a})=\nabla(\nabla\cdot\mathbf{a})-\nabla^2\mathbf{a}$	$\nabla\cdot(\psi\mathbf{a})=\mathbf{a}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{a}$
$\nabla\times(\psi\mathbf{a})=\nabla\psi\times\mathbf{a}+\psi\nabla\times\mathbf{a}$	$\nabla\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b})$
$\nabla(\mathbf{a}\cdot\mathbf{b})=(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a})$	
$\nabla\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}(\nabla\cdot\mathbf{b})-\mathbf{b}(\nabla\cdot\mathbf{a})+(\mathbf{b}\cdot\nabla)\mathbf{a}-(\mathbf{a}\cdot\nabla)\mathbf{b}$	

<u>Vector Differential Operators in Non-Cartesian Coordinates</u>:

Cylindrical Coordinates: (ρ,ϕ,z)

$\nabla\psi=\frac{\partial\psi}{\partial\rho}\hat{\mathbf{a}}_\rho+\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\mathbf{a}}_\phi+\frac{\partial\psi}{\partial z}\hat{\mathbf{a}}_z$
$\nabla\cdot\mathbf{A}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)+\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}$
$\nabla\times\mathbf{A}=\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\hat{\mathbf{a}}_\rho+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\hat{\mathbf{a}}_\phi+\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi)-\frac{\partial A_\rho}{\partial\phi}\right)\hat{\mathbf{a}}_z$
$\nabla^2\psi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}$

Spherical Coordinates: (r,θ,ϕ)

$\nabla\psi=\frac{\partial\psi}{\partial r}\hat{\mathbf{a}}_r+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\mathbf{a}}_\theta+\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\mathbf{a}}_\phi$
$\nabla\cdot\mathbf{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$
$\nabla\times\mathbf{A}=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{\partial A_\theta}{\partial\phi}\right)\hat{\mathbf{a}}_r+\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\hat{\mathbf{a}}_\theta+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right)\hat{\mathbf{a}}_\phi$
$\nabla^2\psi=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial\psi}{\partial\theta}\sin\theta\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$

<u>Solar-System Physical Data</u>:

Earth Mass: 5.98×10^{24} kg	Moon Mass: 7.35×10^{22} kg
Earth Radius: 6.38×10^{6} m	Moon Radius: 1.74×10^{6} m
Mean Earth-Sun Distance: 1.50×10^{11} m	Mean Moon-Earth Distance: 3.85×10^{8} m
Solar Mass: 1.99×10^{30} kg	Solar Radius: 6.96×10^{8} m

Fundamental Constants:

Gravitational constant	$G = 6.674\times10^{-11}$ N·m^2/kg^2
Coulomb constant ($k = 1/4\pi\varepsilon_0$)	$k = 8.9876\times10^{9}$ N·m^2/C^2
Planck's constant ($\hbar = h/2\pi$)	$h = 6.6262\times10^{-34}$ J·s
Boltzmann constant	$k_B = 1.3086\times10^{-23}$ J/K
Speed of light in vacuum	$c = 2.9979\times10^{8}$ m/s
Proton charge	$e = 1.6022\times10^{-19}$ C
Proton rest mass	$m_p = 1.6726\times10^{-27}$ kg
Electron rest mass	$m_e = 9.1096\times10^{-31}$ kg
Avogadro's number	$N_A = 6.0222\times10^{23}$

Trigonometric Identities:

$\sin^2\alpha + \cos^2\alpha = 1$	$1 + \tan^2\alpha = \sec^2\alpha$
$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$	$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\sin 2\alpha = 2\sin\alpha\cos\alpha$	$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$	$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

Pauli Spin Matrices:

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Indefinite integration:

$f(x)$	$\int f(x)dx$		$f(x)$	$\int f(x)dx$
$(1-x^2)^{-1/2}$ $\lvert x\rvert < 1$	$\arcsin x + C$		$(ax^2+c)^{-1/2}$ $a > 0$	$\frac{1}{\sqrt{a}}\ln(x\sqrt{a} + \sqrt{ax^2+c} + C$
$(1+x^2)^{-1}$	$\arctan x + C$		$(ax^2+c)^{-3/2}$	$\frac{x}{c\sqrt{ax^2+c}} + C$

Definite integration:

$\int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$, n is an integer, and $a > 0$

$\int_0^\infty x^{1/2} e^{-ax}dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}, \quad a > 0$

$\int_0^\infty e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{2a}, \quad a > 0$

$\int_0^\infty x^{2n} e^{-a^2x^2}dx = \frac{(2n-1)!!}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}, \quad a > 0$