Department of Physics and Astronomy University of Georgia

August 2007 Written Comprehensive Exam

— Day 1 —

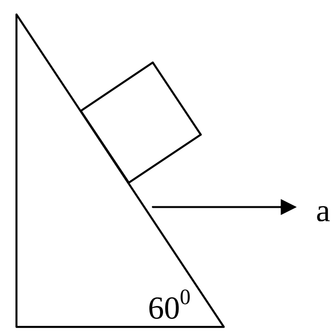
This is a closed-book, closed-note exam. You may use a calculator, but only for arithmetic functions (not for doing algebra or for referring to notes stored in memory). Attempt all six problems. Start ***each problem*** on ***a new sheet of paper*** (not merely on a new side) and use one side only. Print your name on each piece of paper that you submit. For full credit you must show your work and/or explain your answers.

PROBLEM 1 (two parts)

A 2 kg mass rests on a smooth frictionless wedge, which has an inclination of 60^o (see figure below). Suppose that the wedge is accelerating to the right with an acceleration *a* such that the mass remains stationary relative to the wedge.

(a) Find the acceleration *a*.

(b) If the acceleration of the wedge is now doubled, what is the acceleration of the mass along the incline?



PROBLEM 2 (one part)

The point of support of a simple pendulum of mass *m* and length *b* is driven horizontally by $x = a\sin\omega t$. Using Lagrangian techniques, find the pendulum's equation of motion in terms of the angle it makes with the vertical, θ.

PROBLEM 3 (three parts)

(a) State Maxwell's equations.

(b) Show that in the absence of sources the electric field satisfies the wave equation.

(c) By taking the divergence of one of these equations, derive the continuity equation.

PROBLEM 4 (six parts)

In an atom optical interferometer shown below, a highly collimated beam of atoms interacts with three regions of standing-wave laser fields (the intensity of the standing wave modulates the atomic wavefunctions), are created by lasers of wavelength 590 nm.

(a) If the atoms are sodium (A = 23) and derive from a thermal ensemble that has been laser-cooled to a temperature of 1 mK, what is the RMS (root-mean-square) speed of the atoms in the beam?

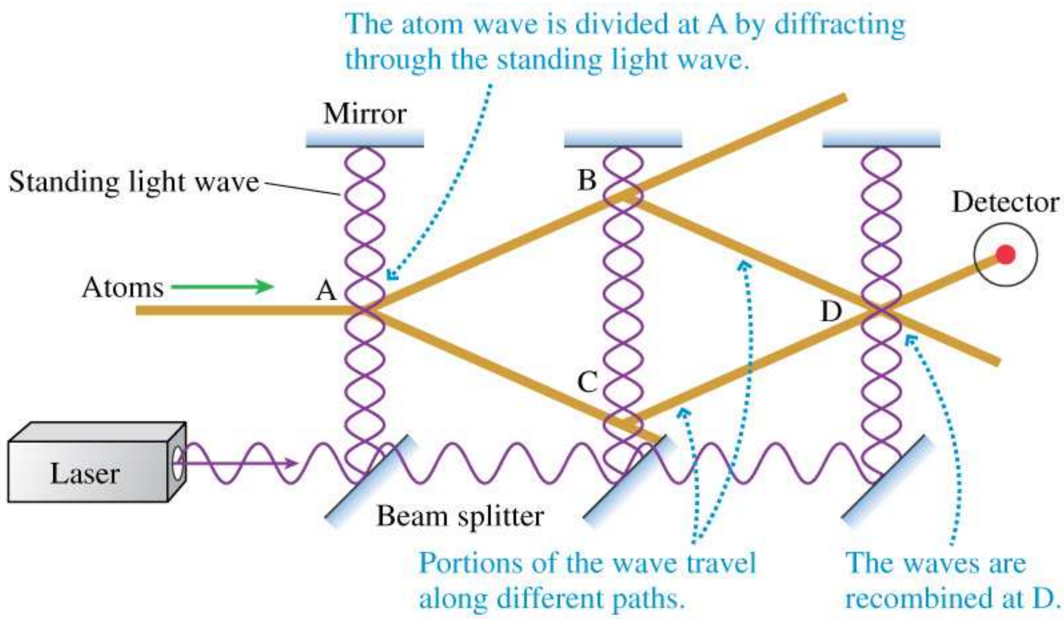
(b) What is the deBroglie wavelength of an atom traveling at the RMS speed?

(c) By treating the laser beam standing wave as a diffraction grating, calculate the first-order diffraction angle of an atom traveling at the RMS speed.

(d) What momentum change does this represent for the atom?

(e) How many 590 nm photon recoil momenta does this momentum change correspond to?

(f) When the diffracted atoms reach the second interaction region, the atoms see the two counter-propagating fields of the standing wave shifted in wavelength by the Doppler effect. Specifically, the atoms arriving at B are moving towards the photons coming from the top mirror, and away from the photons coming from the bottom mirror. The Doppler shifts are equal in magnitude and opposite in sign. What is the magnitude of the shift, in Hz?



PROBLEM 5 (three parts)

A particle of mass m is in a one-dimensional infinite square well of width a centered on the origin. At t = 0, its normalized wave function is $\psi(x,0) = \alpha\phi_1(x) + \beta\phi_2(x)$, where $\phi_1(x)$ and $\phi_2(x)$ are the two lowest energy eigenstates, respectively, and α and β are real. The expectation value of the energy, $\langle E \rangle$, is known to be 3 times the ground-state energy.

(a) Determine α and β.

(b) Determine $\psi(x,t)$ and $|\psi(x,t)|^2$. Express the latter as a sinusoidal function of time. [For simplicity, denote $\omega = \pi^2\hbar / 2ma^2$.]

(c) Show that the time-dependent expectation value of position has the following form:

$$\langle x \rangle_t = A\cos(n\omega t),$$

where ω is as defined in (b), n is an integer, and A is a constant you need not evaluate. Determine the value of n.

PROBLEM 6 (three parts)

A wave has angular frequency ω and wavevector k.

(a) Write down the phase velocity, v, in terms of ω and k.

(b) Write down the group velocity, v_g, in terms of ω and k.

(c) Show that

$$v_g = v + k\frac{dv}{dk}$$

and

$$v_g = v - \lambda\frac{dv}{d\lambda}$$

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— Day 2 —

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PROBLEM 1 (five parts)

The potential energy associated with the force between two identical atoms in a diatomic molecule separated by a distance *r* can be represented approximately by the expression

$$U(r) = U_0\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right]$$

where r_0 and U_0 are constants for that molecule.

(a) Find an expression for the force between the two atoms, F_r.

(b) At what value of *r* is the potential energy zero? At what value of *r* is the force zero?

(c) At what value of *r* is the potential energy a minimum and what is this potential energy?

(d) Make a rough plot of *U*(*r*) versus *r*. (The shape of the curve, the relative positive and negative values should be plot correctly)

(e) If these two atoms each have mass, *m*, find an expression for the vibrational frequency of this diatomic molecule in terms of U_0, r_0, and *m*. Treat the problem as a classical problem like that of two masses connected by a spring. Assume a small amplitude of vibration and note that $(1 + x)^n = 1 + nx$ (for $x \ll 1$).

PROBLEM 2 (one part)

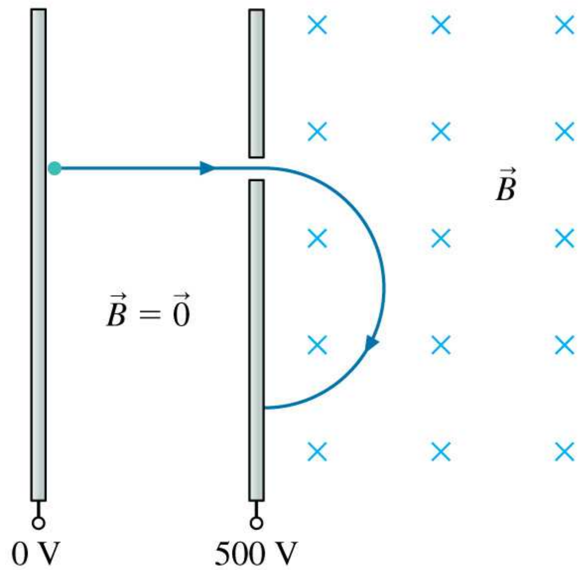
Find the capacitance of two concentric metal shells with radii *a* and *b* where *b* > *a*.
(***Hint***: place a charge −*Q* on the inner sphere and +*Q* on the outer sphere)

PROBLEM 3 (two parts)

A particle of mass m and charge $-e$ is accelerated from rest through a potential difference of 500V, then injected through a slit into a uniform magnetic field of strength 8.91×10^{-3} T. It strikes the second plate 16.94 mm away from the slit, as shown in the figure below.

(a) What is the mass of the particle (in kg)?

(b) If 6.25×10^{12} of these particles per second are absorbed by the second plate, how much cooling power would have to be delivered to the plate to keep it from heating?



PROBLEM 4 (two parts)

An engine using 1 mol of an ideal gas initially at V_1 = 24.6 L and T_1 = 400 K performs a cycle consisting of four steps: (1) isothermal expansion at T_1 to twice its initial volume, $V_2 = 2V_1$; (2) cooling at constant volume to T_2 = 300 K; (3) isothermal compression to its original volume; and (4) heating at constant volume to its original temperature. Assume the heat capacity at constant volume for this gas is C_V = 21 J/K. [The ideal gas constant is R = 8.314 J/(mol·K).]

(a) Sketch the engine cycle on a PV diagram, labeling quantitative values on each axis at important points along the cycle. Make sure each step of the cycle is labeled with its corresponding number on your diagram.

(b) Determine the efficiency of this engine.

PROBLEM 5 (one part)

Find the expectation value of the kinetic energy in the nth state of the one-dimensional quantum harmonic oscillator. (*Please show your derivation of the result.*)

PROBLEM 6 (three parts)

A spin-½ particle of gyromagnetic ratio γ (*i.e.*, magnetic moment operator $\hat{\mu} = \gamma\hat{\mathbf{S}}$) is at rest in an oscillating, spatially uniform magnetic field:

$$\mathbf{B} = B_0 \cos(\omega t)\,\mathbf{e}_z\,,$$

where $\mathbf{e}_z$ is the unit vector in the z direction, and B_0 and ω are positive constants.

(a) Construct the Hamiltonian matrix for this system in the standard up-down basis.

(b) Suppose the particle starts out in the $+\hbar/2$ eigenstate of $\hat{S}_x$; *i.e.* $|\chi, 0\rangle = |S_x +\rangle$. Determine $|\chi, t\rangle$ at any subsequent time. [Be careful. The Hamiltonian is *not* time-independent, so you will have to solve the time-dependent Schrödinger equation.]

(c) Find the probability of obtaining $-\hbar/2$ in a measurement of S_x. What is the minimum field strength, B_0, for which it is possible to completely flip the spin?

Note Sheet for Fall 2007 Qualifying Exam

<u>Vector Identities</u>: (**a**, **b**, **c** and **d** are vector fields and ψ is a scalar field)

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$	$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$
$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{d})$	
$\nabla\times\nabla\psi=0$	$\nabla\cdot(\nabla\times\mathbf{a})=0$
$\nabla\times(\nabla\times\mathbf{a})=\nabla(\nabla\cdot\mathbf{a})-\nabla^2\mathbf{a}$	$\nabla\cdot(\psi\mathbf{a})=\mathbf{a}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{a}$
$\nabla\times(\psi\mathbf{a})=\nabla\psi\times\mathbf{a}+\psi\nabla\times\mathbf{a}$	$\nabla\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b})$
$\nabla(\mathbf{a}\cdot\mathbf{b})=(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a})$	
$\nabla\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}(\nabla\cdot\mathbf{b})-\mathbf{b}(\nabla\cdot\mathbf{a})+(\mathbf{b}\cdot\nabla)\mathbf{a}-(\mathbf{a}\cdot\nabla)\mathbf{b}$	

<u>Vector Differential Operators in Non-Cartesian Coordinates</u>:

Cylindrical Coordinates: (ρ,ϕ,z)

$\nabla\psi=\frac{\partial\psi}{\partial\rho}\hat{\mathbf{a}}_\rho+\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\mathbf{a}}_\phi+\frac{\partial\psi}{\partial z}\hat{\mathbf{a}}_z$
$\nabla\cdot\mathbf{A}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)+\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}$
$\nabla\times\mathbf{A}=\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\hat{\mathbf{a}}_\rho+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\hat{\mathbf{a}}_\phi+\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi)-\frac{\partial A_\rho}{\partial\phi}\right)\hat{\mathbf{a}}_z$
$\nabla^2\psi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}$

Spherical Coordinates: (r,θ,ϕ)

$\nabla\psi=\frac{\partial\psi}{\partial r}\hat{\mathbf{a}}_r+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\mathbf{a}}_\theta+\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\mathbf{a}}_\phi$
$\nabla\cdot\mathbf{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$
$\nabla\times\mathbf{A}=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{\partial A_\theta}{\partial\phi}\right)\hat{\mathbf{a}}_r+\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\hat{\mathbf{a}}_\theta+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right)\hat{\mathbf{a}}_\phi$
$\nabla^2\psi=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial\psi}{\partial\theta}\sin\theta\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$

<u>Solar-System Physical Data</u>:

Earth Mass: 5.98×10^{24} kg	Moon Mass: 7.35×10^{22} kg
Earth Radius: 6.38×10^{6} m	Moon Radius: 1.74×10^{6} m
Mean Earth-Sun Distance: 1.50×10^{11} m	Mean Moon-Earth Distance: 3.85×10^{8} m
Solar Mass: 1.99×10^{30} kg	Solar Radius: 6.96×10^{8} m

Fundamental Constants:

Gravitational constant	$G = 6.673\times10^{-11}$ N·m^2/kg^2
Coulomb constant ($k = 1/4\pi\varepsilon_0$)	$k = 8.9876\times10^{9}$ N·m^2/C^2
Planck's constant ($\hbar = h/2\pi$)	$h = 6.6262\times10^{-34}$ J·s
Boltzmann constant	$k_B = 1.3086\times10^{-23}$ J/K
Speed of light in vacuum	$c = 2.9979\times10^{8}$ m/s
Proton charge	$e = 1.6022\times10^{-19}$ C
Proton rest mass	$m_p = 1.6726\times10^{-27}$ kg
Electron rest mass	$m_e = 9.1096\times10^{-31}$ kg
Avogadro's number	$N_A = 6.0222\times10^{23}$

Trigonometric Identities:

$\sin^2\alpha + \cos^2\alpha = 1$	$1 + \tan^2\alpha = \sec^2\alpha$
$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$	$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\sin 2\alpha = 2\sin\alpha\cos\alpha$	$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$	$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

Pauli Spin Matrices:

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$